Albanese Varieties with Modulus over a Perfect Field

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Abstract

Let X be a smooth proper variety over a perfect field k of arbitrary characteristic. Let D be an effective divisor on X with multiplicity. We introduce an Albanese variety $\mathrm{Alb}(X,D)$ of X of modulus D as a higher dimensional analogon of the generalized Jacobian of Rosenlicht-Serre with modulus for smooth proper curves. Basing on duality of 1-motives with unipotent part (which are introduced here), we obtain explicit and functorial descriptions of these generalized Albanese varieties and their dual functors.

We define a relative Chow group of zero cycles $\mathrm{CH}_0(X,D)$ of modulus D and show that $\mathrm{Alb}(X,D)$ can be viewed as a universal quotient of $\mathrm{CH}_0(X,D)^0$.

As an application we can rephrase Lang's class field theory of function fields of varieties over finite fields in explicit terms.

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0 Introduction

The generalized Jacobian variety with modulus of a smooth proper curve X over a field is a well-established object in algebraic geometry and number theory and turned out to be of great benefit e.g. for the theory of algebraic groups, ramification theory and class field theory. In this work we extend this notion from [Ser3, V] to the situation of a higher dimensional smooth proper variety X over a perfect field k. The basic idea of this construction comes from [Ru] and is accomplished in [KR1], both only for the case that k is of characteristic 0. Positive characteristic however requires distinct methods and turns out to be the difficult part of the story.

To a rational map $\varphi : X \dashrightarrow G$ from X to a commutative algebraic group G we assign an effective divisor $\operatorname{mod}\varphi$, the *modulus of* φ. (Def. 3.11). Our definition from [KR2] coincides with the classical definition in the curve case as in [Ser3, III, No. 1]. For an effective divisor D on X the generalized Albanese variety $\mathrm{Alb}(X,D)$ of X of modulus D and the Albanese map $\mathrm{alb}_{X,D} : X \dashrightarrow \mathrm{Alb}(X,D)$ are defined by the following universal property: for every commutative algebraic group G and every rational map φ from X to G of modulus $\leq D$ there exists a unique homomorphism of algebraic groups $h : \mathrm{Alb}(X,D) \longrightarrow G$ such that $\varphi = h \circ \mathrm{alb}_{X,D}$ up to translation by a constant $g \in G(k)$. Every rational map to a commutative algebraic group admits a modulus, and the effective divisors on X form an inductive system. Then the projective limit $\varprojlim \mathrm{Alb}(X,D)$ over all effective divisors D on X yields a pro-algebraic group that satisfies the universal mapping property for all rational maps from X to commutative algebraic groups.

The Albanese variety with modulus arises as a special case of a broader notion of generalized Albanese varieties defined by a universal mapping property for categories of rational maps from X to commutative algebraic groups (Thm. 0.2). As the construction of these universal objects is based on duality, a notion of duality for smooth connected commutative algebraic groups over a perfect field k of arbitrary characteristic is required. For this purpose we introduce so called *1-motives with unipotent part* (Def. 1.22), which generalize Deligne 1-motives [Del, Définition (10.1.2)] and Laumon 1-motives [Lau, Définition (5.1.1)]. In this context, we obtain explicit and functorial descriptions of these generalized Albanese varieties and their dual functors (Thm. 0.1).

In a geometric way we define a relative Chow group of 0-cycles $\mathrm{CH}_0(X,D)$ with respect to the modulus D (Def. 3.28). Then we can realize $\mathrm{Alb}(X,D)$ as a universal quotient of $\mathrm{CH}_0(X,D)^0$, the subgroup of $\mathrm{CH}_0(X,D)$ of cycles of degree 0 (Thm. 0.3). The relation of $\mathrm{CH}_0(X,D)$ to the K-theoretic idéle

class groups from [KS] gives rise to some future study, but is beyond the scope of this paper. Using these idéle class groups, Önsiper [Ön] proved the existence of generalized Albanese varieties for smooth proper surfaces in characteristic $p > 0$.

Lang's class field theory of function fields of varieties over finite fields [Ser3, V] is written in terms of so called *maximal maps*, which appeared as a purely theoretical notion, apart from their existence very little seemed to be known about which. The Albanese map with modulus allows to replace these black boxes by concrete objects (Thm. 0.4).

We present the main results by giving a summary of each section.

0.1 Leitfaden

Section 1 is devoted to the following generalization of 1-motives: A *1-motive with unipotent part* (Definition 1.22) is roughly a homomorphism $[\mathcal{F} \to G]$ in the category of sheaves of abelian groups over an algebraically closed field k from a dual-algebraic commutative formal group $\mathcal{F}$ to an extension G of an abelian variety A by a commutative affine algebraic group L. Here a commutative formal group $\mathcal{F}$ is called *dual-algebraic* if its Cartier-dual $\mathcal{F}^\vee = \underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{G}_\mathrm{m})$ is algebraic. 1-motives with unipotent part admit duality (No. 1.1.7). The dual of $[0 \to G]$ is given by $[L^\vee \to A^\vee]$, where $L^\vee = \underline{\mathrm{Hom}}(L, \mathbb{G}_\mathrm{m})$ is the Cartier-dual of L and $A^\vee = \mathrm{Pic}^0_A = \underline{\mathrm{Ext}}(A, \mathbb{G}_\mathrm{m})$ is the dual abelian variety of A, and the homomorphism between them is the connecting homomorphism associated to $0 \to L \to G \to A \to 0$. In particular, every smooth connected commutative algebraic group over k has a dual in this category. Moreover, these 1-motives may contain torsion.

Section 2. In the framework of *categories of rational maps from a smooth proper variety X over an algebraically closed field k to commutative algebraic k-groups* (Definition 2.7), we ask for the existence of universal objects (Definition 2.11) for such categories, i.e. objects having the universal mapping property with respect to the category they belong to. A necessary and sufficient condition for the existence of such universal objects is given in Theorem 2.12, as well as their explicit construction, using duality of 1-motives with unipotent part. (This was done in [Ru] for the case $\mathrm{char}(k) = 0$.)

In particular we show the following: Let $\underline{\mathrm{Div}}_X$ be the sheaf of relative Cartier divisors, i.e. the sheaf of abelian groups that assigns to any k-algebra R the group $\underline{\mathrm{Div}}_X(R)$ of all Cartier divisors on $X \otimes_k R$ generated locally on $\mathrm{Spec}\, R$ by effective divisors which are flat over R. Let $\underline{\mathrm{Pic}}_X$ be the Picard functor and $\mathrm{Pic}^{0,\mathrm{red}}_X$ the Picard variety of X. Then let $\underline{\mathrm{Div}}^{0,\mathrm{red}}_X$ be the inverse

image of $\mathrm{Pic}_X^{0,\mathrm{red}}$ under the class map $\mathrm{cl} : \underline{\mathrm{Div}}_X \longrightarrow \underline{\mathrm{Pic}}_X$. A rational map $\varphi : X \dashrightarrow G$, where G is a smooth connected commutative algebraic group with affine part L, induces a natural transformation $\tau_\varphi : L^\vee \longrightarrow \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ (No. 2.2.1). If $\mathcal{F}$ is a formal subgroup of $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$, denote by $\mathbf{Mr}_\mathcal{F}$ the category of rational maps for which the image of this induced transformation lies in $\mathcal{F}$.

Theorem 0.1. *Let $\mathcal{F}$ be a dual-algebraic formal subgroup of $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. The category $\mathbf{Mr}_\mathcal{F}$ admits a universal object $\mathrm{alb}_\mathcal{F} : X \dashrightarrow \mathrm{Alb}_\mathcal{F}(X)$. The algebraic group $\mathrm{Alb}_\mathcal{F}(X)$ arises as an extension of the classical Albanese $\mathrm{Alb}(X)$ by the Cartier-dual of $\mathcal{F}$; it is dual to the 1-motive $\left[\mathcal{F} \longrightarrow \mathrm{Pic}_X^{0,\mathrm{red}}\right]$, the homomorphism induced by the class map $\mathrm{cl} : \underline{\mathrm{Div}}_X \longrightarrow \underline{\mathrm{Pic}}_X$.*

Theorem 0.1 results from the stronger Theorem 2.12, which says roughly that a category of rational maps admits a universal object if and only if it is of the shape $\mathbf{Mr}_\mathcal{F}$ for some dual-algebraic formal subgroup $\mathcal{F}$ of $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$.

The generalized Albanese varieties $\mathrm{Alb}_\mathcal{F}(X)$ satisfy an obvious functoriality property (No. 2.3.2). A Galois descent allows to carry over the results to perfect base fields which are not necessarily algebraically closed (No. 2.3.3).

Section 3 is the main part of this work, where we establish a higher dimensional analogon to the generalized Jacobian with modulus of Rosenlicht-Serre. Let X be a smooth proper variety. We use the notion of modulus from [KR2], which associates to a rational map $\varphi : X \dashrightarrow G$ an effective divisor $\mathrm{mod}(\varphi)$ on X (Definition 3.11). If D is an effective divisor on X, we define a formal subgroup $\mathcal{F}_{X,D} = (\mathcal{F}_{X,D})_{\text{ét}} \times_k (\mathcal{F}_{X,D})_{\mathrm{inf}}$ of $\underline{\mathrm{Div}}_X$ (cf. Definition 3.13) by the conditions

$$(\mathcal{F}_{X,D})_{\text{ét}} = \left\{ B \in \underline{\mathrm{Div}}_X(k) \;\middle|\; \mathrm{Supp}(B) \subset \mathrm{Supp}(D) \right\}$$

and if $\mathrm{char}(k) = 0$

$$(\mathcal{F}_{X,D})_{\mathrm{inf}} = \exp\left(\widehat{\mathbb{G}}_{\mathrm{a}} \otimes_k \Gamma\big(X, \mathcal{O}_X\left(D - D_{\mathrm{red}}\right) / \mathcal{O}_X\big)\right)$$

if $\mathrm{char}(k) = p > 0$

$$(\mathcal{F}_{X,D})_{\mathrm{inf}} = \mathrm{Exp}\left(\sum_{r>0} {}_r\widehat{\mathbb{W}} \otimes_{\mathbb{W}(k)} \Gamma\Big(X, \mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}} \mathbb{W}_r(\mathcal{K}_X) \Big/ \mathbb{W}_r(\mathcal{O}_X)\Big)\right)$$

where D_{red} is the underlying reduced divisor of D, Exp denotes the Artin-Hasse exponential and $\mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)$ is a filtration of the Witt group (Definition 3.2). Let $\mathcal{F}_{X,D}^{0,\mathrm{red}} = \mathcal{F}_{X,D} \times_{\underline{\mathrm{Div}}_X} \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ be the intersection of $\mathcal{F}_{X,D}$ and

$\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. The formal groups $\mathcal{F}_{X,D}$ and $\mathcal{F}_{X,D}^{0,\mathrm{red}}$ are dual-algebraic (Proposition 3.14).

Then it holds $\mathrm{mod}(\varphi) \leq D$ if and only if $\mathrm{im}(\tau_\varphi) \subset \mathcal{F}_{X,D}^{0,\mathrm{red}}$ (Lemma 3.17). This yields (cf. Theorem 3.19)

Theorem 0.2. *The category* $\mathbf{Mr}^{X,D}$ *of those rational maps* $\varphi : X \dashrightarrow G$ *s.t.* $\mathrm{mod}(\varphi) \leq D$ *admits a universal object* $\mathrm{alb}_{X,D} : X \dashrightarrow \mathrm{Alb}(X,D)$*, called the* Albanese of X of modulus D. *The algebraic group* $\mathrm{Alb}(X,D)$ *is dual to the 1-motive* $\left[\mathcal{F}_{X,D}^{0,\mathrm{red}} \longrightarrow \mathrm{Pic}_X^{0,\mathrm{red}}\right]$.

The Albanese varieties with modulus $\mathrm{Alb}(X,D)$ are functorial (No. 3.2.2) and descend to arbitrary perfect base field (No. 3.2.3).

In the case that $X = C$ is a curve, our Albanese with modulus $\mathrm{Alb}(C,D)$ coincides with the generalized Jacobian with modulus $J(C,D)$ of Rosenlicht-Serre (Theorem 3.26).

A relative Chow group $\mathrm{CH}_0(X,D)$ of modulus D is introduced in Definition 3.28. We say a rational map $\varphi : X \dashrightarrow G$ to a commutative algebraic group G *factors through* $\mathrm{CH}_0(X,D)^0$ if the associated map on 0-cycles of degree 0 (where U is the open set on which φ is defined) $\mathrm{Z}_0(U)^0 \longrightarrow G(k)$, $\sum l_i\, p_i \longmapsto \sum l_i\, \varphi(p_i)$ factors through a homomorphism of abstract groups $\mathrm{CH}_0(X,D)^0 \longrightarrow G(k)$. We show (cf. Theorem 3.30)

Theorem 0.3. *A rational map* $\varphi : X \dashrightarrow G$ *factors through* $\mathrm{CH}_0(X,D)^0$ *if and only if it factors through* $\mathrm{Alb}(X,D)$ *modulo translation. In other words,* $\mathrm{Alb}(X,D)$ *is a universal quotient of* $\mathrm{CH}_0(X,D)^0$.

The theory of Albanese varieties with modulus has an application to the class field theory of function fields of varieties over finite fields. Let X be a geometrically irreducible projective variety over a finite field $k = \mathbb{F}_q$. Let $\overline{k}$ be an algebraic closure of k. Let K_X denote the function field of X, let $\mathrm{K}_X^{\mathrm{ab}}$ be the maximal abelian extension of K_X. From Lang's class field theory one obtains (cf. Theorem 3.36)

Theorem 0.4. *The geometric Galois group* $\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} \,/\, \mathrm{K}_X\, \overline{k}\right)$ *is isomorphic to the projective limit of the* k*-rational points of the Albanese varieties of* X *with modulus* D

$$\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} \,/\, \mathrm{K}_X\, \overline{k}\right) \cong \varprojlim_D \mathrm{Alb}(X,D)(k)$$

where D *ranges over all effective divisors on* X *rational over* k.

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1 1-Motives

1.1 Algebraic Groups and Formal Groups

In this subsection we recall some basics on group functors, algebraic groups and formal groups which are fundamental for 1-motives with unipotent part. References for algebraic groups are [DG] and [Wat], for formal groups and Cartier duality are [SGA3, $\mathrm{VII_B}$], [Dem, II] and [Fon, I]. A key-tool for our description of Cartier duality will be the functor $R \longmapsto \mathbb{L}_R$ (No. 1.1.6), which assigns to a k-algebra R the Weil restriction $\mathbb{L}_R := \Pi_{R/k}\mathbb{G}_{\mathrm{m},R}$ of $\mathbb{G}_{\mathrm{m},R}$ from R to k.

1.1.1 Group Functors

Let k be a ring (i.e. associative, commutative and with unit). Let **Set** be the category of sets. Let **Ab** be the category of abelian groups. Let $\mathbf{Alg}/k$ be the category of k-algebras, and let $\mathbf{Art}/k$ be the category of finite k-algebras (i.e. of finite length). Let $\mathfrak{C}$ be an arbitrary category.

A *k-functor (with values in $\mathfrak{C}$)* is by definition a covariant functor from $\mathbf{Alg}/k$ to $\mathfrak{C}$. The category of k-functors is denoted by $\mathbf{Fctr}(\mathbf{Alg}/k, \mathfrak{C})$. The morphisms are given by natural transformations of functors.

A *formal k-functor (with values in $\mathfrak{C}$)* is by definition a covariant functor from $\mathbf{Art}/k$ to $\mathfrak{C}$. The category of formal k-functors is denoted by $\mathbf{Fctr}(\mathbf{Art}/k, \mathfrak{C})$.

The inclusion $\mathbf{Art}/k \longrightarrow \mathbf{Alg}/k$ induces the *completion functor* $\widehat{?} : \mathbf{Fctr}(\mathbf{Alg}/k, \mathfrak{C}) \longrightarrow \mathbf{Fctr}(\mathbf{Art}/k, \mathfrak{C})$. The completion $\widehat{F}$ of a k-functor F is given by $\widehat{F}(R) = F(R)$ for $R \in \mathbf{Art}/k$.

A (formal) k-functor with values in **Ab** is called a *(formal) k-group functor*. The category of k-group functors will be denoted by $\mathbf{Ab}/k$ (rather than $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$).

1.1.2 Base Extension and Weil Restriction

Let S be a k-algebra.

Base extension is the functor $?_S : \mathbf{Fctr}(\mathbf{Alg}/k, \mathfrak{C}) \longrightarrow \mathbf{Fctr}(\mathbf{Alg}/S, \mathfrak{C})$, $F \longmapsto F_S$ defined by $F_S(R) = F({}_kR)$ for any S-algebra R, where ${}_kR$ is the underlying k-algebra. F_S is sometimes also denoted by $F \otimes_k S$.

Weil restriction is the functor $\Pi_{S/k} : \mathbf{Fctr}(\mathbf{Alg}/S, \mathfrak{C}) \longrightarrow \mathbf{Fctr}(\mathbf{Alg}/k, \mathfrak{C})$, $G \longmapsto G(? \otimes_k S)$.

Weil restriction is the right adjoint functor to base extension, i.e. for $F \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathfrak{C})$ and $G \in \mathbf{Fctr}(\mathbf{Alg}/S, \mathfrak{C})$ it holds $\mathrm{Hom}_{\mathcal{A}b/S}\big(F_S, G\big) =$

$\mathrm{Hom}_{\mathcal{A}b/k}\big(F, G\,(?\otimes S)\big)$ (see [DG, I, § 1, 6.6]).

A k-functor F with values in **Set** is said to be *represented by a k-scheme* if there is a scheme X over k and functorial (in $R \in \mathbf{Alg}/k$) isomorphisms $F(R) \cong \mathrm{Mor}_k(\mathrm{Spec}\, R, X)$.

Proposition 1.1. *Let S be a k-algebra which is a finite projective k-module. Let G be an S-functor with values in* **Set**. *If G is represented by an affine S-scheme X, then its Weil restriction $G\,(?\otimes S)$ is represented by an affine k-scheme $\Pi_{S/k}X$. Moreover, if X is separated and of finite type over S, then $\Pi_{S/k}X$ is separated and of finite type over k.*

Proof. [DG, I, § 1, 6.6] or [BLR, 7.6, Thm. 4 and its proof]. ■

1.1.3 Algebraic Groups

From now on, k is assumed to be a field, and $\overline{k}$ an algebraic closure of k.

A *k-group* (or *k-group scheme*) is by definition a k-group functor with values in **Ab** (we only consider commutative group schemes) whose underlying set-valued k-functor is represented by a k-scheme. The category of k-groups is denoted by $\mathcal{G}/k$, the category of affine k-groups by $\mathcal{G}a/k$.

An *algebraic k-group* is a k-group whose underlying scheme is separated and of finite type over k. The category of algebraic k-groups is denoted by $a\mathcal{G}/k$, the category of affine algebraic k-groups by $a\mathcal{G}a/k$. If there is no confusion about the base field k, we write just *algebraic group*.

Theorem 1.2. *The categories $\mathcal{G}a/k$ of affine k-groups, $a\mathcal{G}/k$ of algebraic k-groups and $a\mathcal{G}a/k$ of affine algebraic k-groups are abelian.*

Proof. See [Dem, II, No. 6, p. 30] for $\mathcal{G}a/k$, [SGA3, $\mathrm{VI_A}$, 5.4, p. 315] for $a\mathcal{G}/k$. The statement for $a\mathcal{G}a/k$ follows from this, since $a\mathcal{G}a/k$ is the intersection of $\mathcal{G}a/k$ and $a\mathcal{G}/k$ in $\mathcal{G}/k$. ■

A k-group L is called *multiplicative* if it is affine and if over an algebraic closure $\overline{k}$ of k it holds that $L \otimes_k \overline{k}$ is the spectrum of the group-algebra $\overline{k}[\Gamma]$ of some constant group Γ: $L \otimes_k \overline{k} = \mathrm{Spec}\,\overline{k}[\Gamma]$.

A k-group L is called *unipotent* if it is affine and if for each non-trivial closed subgroup H of L there exists a non-trivial homomorphism of k-groups $H \longrightarrow \mathbb{G}_{\mathrm{a},k}$.

Theorem 1.3. *An affine k-group L is canonically an extension of a unipotent group U by a multiplicative group M. If the base field k is perfect, this extension splits canonically, i.e. there is a unique isomorphism $L \cong M \times_k U$.*

Proof. [DG, IV, § 3, 1.1]. ■

A *torus* T of dimension d defined over k is by definition an algebraic k-group with the following property: there exists a finite field extension k_1/k such that $T \otimes_k k_1$ splits into a direct product of copies of $\mathbb{G}_{\mathrm{m}}$.

An affine algebraic k-group L is unipotent if and only if there exists a faithful representation of L as a group of unipotent matrices in some GL_r. Every unipotent algebraic group is isomorphic to a closed subgroup of the strict upper triangular group (see [Wat, 8.3 Thm. p. 64]). A smooth connected unipotent group U has the property: there exists a finite field extension k_1/k such that the underlying k_1-scheme of $U \otimes_k k_1$ is isomorphic to some affine scheme $\mathbb{A}^s_{k_1}$ over k_1 (see [Ser3, VII, No. 6, Cor. of Prop. 7]).

Theorem 1.4. *A smooth connected affine algebraic k-group L is canonically an extension of a smooth connected unipotent algebraic k-group U by a k-torus T. If the base field k is perfect, this extension splits canonically, i.e. there is a unique isomorphism $L \cong T \times_k U$.*

Proof. [SGA3, XVII, 7.2.1]. ■

Theorem 1.5 (Chevalley)**.** *A smooth connected algebraic k-group G admits a* canonical decomposition $0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$, *where L is a connected affine algebraic k-group and A is an abelian k-variety. If the base field k is perfect, L is smooth.*

Proof. [Bar, Thm. 3.2, p. 97] or [Ros, No. 5, Thm. 16, p. 439] or [Che]. ■

1.1.4 Formal Groups

Let k be a field.

A *formal k-scheme* is by definition a formal k-functor with values in **Set** which is the limit of a directed inductive system of finite k-schemes:

A formal k-functor $\mathcal{F}$ is represented by a formal k-scheme if there exists a directed projective system (A_i) of finite k-rings and an isomorphism of functors $\mathcal{F} \cong \varinjlim \operatorname{Spec} A_i$.

Definition 1.6. Let $\mathcal{A}$ be a *profinite* k-algebra, i.e. a complete topological k-algebra whose topology has a basis of neighbourhoods of zero consisting of ideals of finite codimension; this means $\mathcal{A}$ is the projective limit (as a topological ring) of discrete quotients which are finite k-algebras.

The *formal spectrum of A* is by definition the formal k-functor which assigns to $R \in \mathbf{Art}/k$ the set of continuous homomorphisms of k-algebras from the topological ring $\mathcal{A}$ to the discrete ring R: $\operatorname{Spf}\mathcal{A}\,(R) = \operatorname{Hom}^{\text{cont}}_{k\text{-alg}}(\mathcal{A}, R)$.

Proposition 1.7. *For a formal k-functor $\mathcal{F}$ the following conditions are equivalent:*

(i) *$\mathcal{F}$ is represented by a formal k-scheme.*

(ii) *There is a profinite k-algebra $\mathcal{A}$ and an isomorphism $\mathcal{F} \cong \operatorname{Spf}\mathcal{A}$.*

(iii) *$\mathcal{F}$ is left-exact, i.e. $\mathcal{F}$ commutes with finite projective limits.*

(See [Dem, I, No. 6] or [Fon, I, § 4].)

A *formal k-group* is a formal k-group functor with values in $\mathbf{Ab}$ (we are only interested in commutative formal groups) whose underlying set-valued formal k-functor is represented by a formal scheme over k. The category of k-formal groups is denoted by $\mathcal{G}f/k$. If there is no confusion about the base field k, we write just *formal group.*

Remark 1.8. A formal k-group $\mathcal{F} : \mathbf{Art}/k \longrightarrow \mathbf{Ab}$ extends in a natural way to a k-group functor $\widetilde{\mathcal{F}} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$, by defining $\widetilde{\mathcal{F}}(R)$ for $R \in \mathbf{Alg}/k$ as the inductive limit of the $\mathcal{F}(S)$, where S ranges over the finite k-subalgebras of R. If $\mathcal{F} = \operatorname{Spf}\mathcal{A}$ for some profinite k-algebra $\mathcal{A}$, then $\widetilde{\mathcal{F}}(R) = \operatorname{Hom}^{\text{cont}}_{k\text{-alg}}(\mathcal{A}, R)$ for every $R \in \mathbf{Alg}/k$.

Theorem 1.9. *The category $\mathcal{G}f/k$ of formal k-groups is abelian.*

Proof. [SGA3, VII_B, 2.4.2, p. 521]. ■

(This is equivalent to the fact that the category $\mathcal{G}a/k$ is abelian, by Cartier-duality, see 1.1.7.)

Theorem 1.10. *A formal k-group $\mathcal{F}$ is canonically an extension of an étale formal k-group $\mathcal{F}_{\text{ét}}$ by a connected (= infinitesimal) formal k-group (i.e. the formal spectrum of a local ring) $\mathcal{F}_{\text{inf}}$. Here $\mathcal{F}_{\text{ét}}\,(R) = \mathcal{F}\,(R_{\text{red}})$ and $\mathcal{F}_{\text{inf}}\,(R) = \ker\Big(\mathcal{F}\,(R) \longrightarrow \mathcal{F}\,(R_{\text{red}})\Big)$ for $R \in \mathbf{Art}/k$, $R_{\text{red}} = R/\operatorname{Nil}(R)$. If the base field k is perfect, there is a unique isomorphism $\mathcal{F} \cong \mathcal{F}_{\text{inf}} \times_k \mathcal{F}_{\text{ét}}$.*

Proof. [Dem, I, No. 7, Prop. on p. 34] or [Fon, I, 7.2, p. 46]. ■

1.1.5 Sheaves of Abelian Groups

Let R be a ring. Let $\mathfrak{C}$ be an arbitrary category.

An *R-sheaf (with values in $\mathfrak{C}$)* is a sheaf (of objects of $\mathfrak{C}$) on $\mathbf{Alg}/R$ for the topology fppf. An R-sheaf with values in $\mathbf{Ab}$ is called an *R-group sheaf.*

The category of R-group sheaves is denoted by $\mathcal{A}b/R$. The category $\mathcal{A}b/R$ is by definition a full subcategory of the category of R-group functors $\mathbf{Ab}/R$.

Let k be a field. The category of k-groups $\mathcal{G}/k$ and the category of formal k-groups $\mathcal{G}f/k$ are full subcategories of $\mathcal{A}b/k$.

1.1.6 Linear Group associated to a Ring

Let k be a field.

Definition 1.11. Let R be a k-algebra. The *linear group associated to* R is the k-group sheaf $\mathbb{L}_R = \mathbb{G}_{\mathrm{m}}(? \otimes R)$.

If S is a finite k-algebra, then $\mathbb{L}_S$ is an affine algebraic k-group, according to Proposition 1.1. Thus the completion of the functor $\mathbb{L}_? : \mathbf{Alg}/k \longrightarrow \mathcal{A}b/k$ is a formal k-group functor with values in $a\mathcal{G}a/k$ (we omit the $\widehat{}$ here):

$$\mathbb{L}_? \; : \; \mathbf{Art}/k \longrightarrow a\mathcal{G}a/k.$$

Suppose now that the base field k is perfect. Every finite k-algebra S is of the form $S = S_{\text{ét}} \oplus \operatorname{Nil}(S)$, where $S_{\text{ét}} \cong S_{\text{red}} = S/\operatorname{Nil}(S)$. Since $\mathbb{G}_{\mathrm{m}}$ is left-exact and tensor-product over a field k is exact, the linear group functor $\mathbb{L}_?$ is left-exact.[1] This yields a splitting

$$\mathbb{L}_S = \mathbb{T}_S \times_k \mathbb{U}_S$$

where $\mathbb{T}_S = \mathbb{G}_{\mathrm{m}}(? \otimes S_{\text{red}})$ and $\mathbb{U}_S = \ker\Big(\mathbb{G}_{\mathrm{m}}(? \otimes S) \longrightarrow \mathbb{G}_{\mathrm{m}}(? \otimes S_{\text{red}})\Big)$. One can show that $\mathbb{T}_S$ is a torus over k and $\mathbb{U}_S$ is a smooth connected unipotent algebraic k-group.

Lemma 1.12. *Let k be an algebraically closed field. Every affine algebraic k-group L is isomorphic to a closed subgroup of $\mathbb{L}_S$ for some $S \in \mathbf{Art}/k$.*

Proof. Every affine algebraic k-group L is isomorphic to a closed subgroup of GL_r for some $r \in \mathbb{N}$ (see [Wat, 3.4 Thm. p. 25]). Let $\rho : L \longrightarrow \mathrm{GL}_r$ be a faithful representation. Define S to be the group algebra of $\rho(L)$, i.e the k-subalgebra of the algebra of $(r \times r)$-matrices $\mathrm{Mat}_{r\times r}(k)$ generated by $\rho(L)(k)$. In particular, S is finite dimensional. Here we may assume that L is reduced, hence determined by its k-valued points: otherwise embed the multiplicative part into $(\mathbb{G}_{\mathrm{m}})^t$ for some $t \in \mathbb{N}$ (see [DG, IV, § 1, 1.5]) and the unipotent part into $(\mathbb{W}_r)^n$ for some $r, n \in \mathbb{N}$ (see [DG, V, § 1, 2.5]), and replace L by $(\mathbb{G}_{\mathrm{m}})^t \times_k (\mathbb{W}_r)^n$. Then $\rho(L)(k)$ is contained in the unit group of S, and $\rho : L \longrightarrow \mathbb{G}_{\mathrm{m}}(? \otimes S) = \mathbb{L}_S$ is a monomorphism from L to $\mathbb{L}_S$. ∎

[1] Accepting the Full Embedding Theorem [Fre, Chp. 7, Thm. 7.14] one can rephrase this by saying "$\mathbb{L}_?$ is a formal group with values in $a\mathcal{G}a/k$".

1.1.7 Cartier Duality

Let k be a perfect field. We will use the functorial description of Cartier-duality as in [Dem, II, No. 4]. We may consider formal groups as objects of $\mathcal{A}b/k$, cf. Remark 1.8. Let G be a k-group sheaf. Let $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(G, \mathbb{G}_{\mathrm{m}})$ be the k-group sheaf defined by $R \longmapsto \mathrm{Hom}_{\mathcal{A}b/R}(G_R, \mathbb{G}_{\mathrm{m},R})$, which assigns to a k-algebra R the group of homomorphisms of R-group sheaves from G_R to $\mathbb{G}_{\mathrm{m},R}$.

Theorem 1.13. *If G is an affine group (resp. formal group), the k-group sheaf $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(G, \mathbb{G}_{\mathrm{m}})$ is represented by a formal group (resp. affine group) $G^\vee$, which is called the* Cartier dual *of G.*
Cartier duality is an anti-equivalence between the category of affine groups $\mathcal{G}a/k$ and the category of formal groups $\mathcal{G}f/k$. The functors $L \longmapsto L^\vee$ and $\mathcal{F} \longmapsto \mathcal{F}^\vee$ are quasi-inverse to each other.

Proof. See [Dem, II, No. 4, Thm. p. 27] or [Fon, I, 5.4, p. 37] or [SGA3, $\mathrm{VII_B}$, 2.2.2]. ∎

Lemma 1.14. *Let L be an affine group and R a k-algebra. The R-valued points of the Cartier-dual of L are given by*

$$L^\vee(R) \;=\; \mathrm{Hom}_{\mathcal{A}b/k}(L, \mathbb{L}_R)\,.$$

Proof. The statement is due to the fact that Weil restriction is right-adjoint to base extension:
$L^\vee(R) = \mathrm{Hom}_{\mathcal{A}b/R}(L_R, \mathbb{G}_{\mathrm{m},R}) = \mathrm{Hom}_{\mathcal{A}b/k}\big(L, \mathbb{G}_{\mathrm{m},R}(\,?\otimes R)\big)$. ∎

Cartier-Dual of a Multiplicative Group

Proposition 1.15. *Let L be an affine k-group. Then L is multiplicative if and only if the Cartier-dual $L^\vee$ is an étale formal k-group.*

Proof. [Dem, II, No. 8]. ∎

In particular, the Cartier-dual of a split torus $T \cong (\mathbb{G}_{\mathrm{m}})^t$ is a lattice of the same rank: $T^\vee \cong \mathbb{Z}^t$, i.e. a torsion-free étale formal group.

Proposition 1.16. *Let $\mathcal{E}$ be an étale formal k-group. Then the multiplicative k-group $\mathcal{E}^\vee$ is algebraic if and only if $\mathcal{E}(\overline{k})$ is of finite type.*

Proof. [DG, IV, § 1, 1.2]. ∎

Cartier-Dual of a Unipotent Group

Proposition 1.17. *Let L be an affine k-group. Then L is unipotent if and only if the Cartier-dual $L^\vee$ is an infinitesimal formal k-group.*

Proof. [Dem, II, No. 9]. ∎

Proposition 1.18. *Suppose* $\mathrm{char}(k) = p > 0$. *Let L be an affine k-group. The following conditions are equivalent:*

(i) L *is unipotent algebraic.*

(ii) *There is a monomorphism* $L \hookrightarrow (\mathbb{W}_r)^n$ *for some* $r, n \in \mathbb{N}$.

(iii) *There is an epimorphism* $\big({}_r\widehat{\mathbb{W}}\big)^n \twoheadrightarrow L^\vee$ *for some* $r, n \in \mathbb{N}$.

Here $\mathbb{W}$ denotes the k-group of Witt-vectors, $\mathbb{W}_n$ the k-group of Witt-vectors of finite length r; $\widehat{\mathbb{W}}$ is the subfunctor of $\mathbb{W}$ that associates to $R \in \mathbf{Alg}/k$ the set of $(w_0, w_1, \ldots) \in \mathbb{W}(R)$ such that $w_\nu \in \mathrm{Nil}(R)$ for all $\nu \in \mathbb{N}$ and $w_\nu = 0$ for almost all $\nu \in \mathbb{N}$, finally ${}_r\widehat{\mathbb{W}} = \ker\big(\mathrm{F}^r : \widehat{\mathbb{W}} \longrightarrow \widehat{\mathbb{W}}^{(p^r)}\big)$ is the kernel of the r^{th}-power of the Frobenius F.

Proof. (i)$\Longrightarrow$(ii) [DG, V, § 1, 2.5].

(ii)$\Longrightarrow$(i) The underlying k-scheme of $\mathbb{W}_r$ is the affine space $\mathbb{A}^r$, thus $\mathbb{W}_r$ is algebraic. $0 = \mathbb{W}_0 \subset \mathbb{W}_1 \subset \mathbb{W}_2 \subset \ldots \subset \mathbb{W}_r$ is a filtration of $\mathbb{W}_r$ with quotients $\mathbb{W}_\nu/\mathbb{W}_{\nu-1} \cong \mathbb{W}_1 = \mathbb{G}_{\mathrm{a}}$, hence $\mathbb{W}_r$ is unipotent, according to [DG, IV, § 2, 2.5]. Products of unipotent groups and closed subgroups of a unipotent group are unipotent by [DG, IV, § 2, 2.3]. Since L is isomorphic to a closed subgroup of $(\mathbb{W}_r)^n$, it is unipotent and algebraic.

(ii)$\Longleftrightarrow$(iii) This is due to Cartier-duality, the Cartier-dual of $\mathbb{W}_n$ is isomorphic to ${}_n\widehat{\mathbb{W}}$, see [DG, V, § 4, 4.5]. ∎

1.1.8 Dual Abelian Variety

Let k be an algebraically closed field. Let A be an abelian variety over k. The dual of A is given by $A^\vee = \mathrm{Pic}^0 A$. According to the generalized Barsotti-Weil formula (see [Oort, III.18]), the dual abelian variety $A^\vee$ represents the k-group sheaf $\underline{\mathrm{Ext}}_{\mathcal{A}b/k}(A, \mathbb{G}_{\mathrm{m}})$, associated to $R \longmapsto \mathrm{Ext}_{\mathcal{A}b/R}(A_R, \mathbb{G}_{\mathrm{m},R})$.

Let S be a finite k-algebra. $\mathrm{Ext}_{\mathcal{A}b/S}(A_S, \mathbb{G}_{\mathrm{m},S})$ may be identified with the set of primitive elements in $\mathrm{H}^1\big(A_S, \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{A_S})\big)$ by [Oort, III.17.6], and $\mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_S)$ is the set of primitive elements in $\mathrm{H}^1\big(A, \mathbb{L}_S(\mathcal{O}_A)\big)$ by [Ser3, VII, No. 15, Thm. 5]. The elements of $\mathrm{H}^1\big(A_S, \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{A_S})\big)$ are isomorphism classes of $\mathbb{G}_{\mathrm{m}}$-bundles over $A \otimes S$, while elements of $\mathrm{H}^1\big(A, \mathbb{L}_S(\mathcal{O}_A)\big)$ are isomorphism classes of $\mathbb{L}_S$-bundles over A. Each open cover of $A \otimes S$ can be refined to a cover $\{U_\alpha \otimes S\}_\alpha$ induced by an open cover $\{U_\alpha\}_\alpha$ of A.

Both types of principal fiber bundles are determined by a family of transition functions $\{\tau_{\alpha\beta}\}_{\alpha\beta}$ with $\tau_{\alpha\beta} \in \mathbb{G}_{\mathrm{m}}\big(\mathcal{O}_A(U_{\alpha\beta}) \otimes_k S\big) = \mathbb{L}_S\big(\mathcal{O}_A(U_{\alpha\beta})\big)$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Thus there is a canonical identification of the groups $\mathrm{H}^1\big(A_S, \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{A_S})\big)$ and $\mathrm{H}^1\big(A, \mathbb{L}_S(\mathcal{O}_A)\big)$, which obviously preserves primitive elements. We obtain

Proposition 1.19. *Let A be an abelian k-variety and S a finite k-algebra. There is a canonical identification*

$$\Xi_{S/k} : \mathrm{Ext}_{\mathcal{A}b/S}\,(A_S, \mathbb{G}_{\mathrm{m},S}) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_S)$$

induced by Weil restriction of the fibres.
Thus the S-valued points of the dual abelian variety are given by

$$A^\vee(S) \;=\; \mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_S).$$

1.2 1-Motives with Unipotent Part

Let k be an algebraically closed field.

1.2.1 Definition of a 1-Motive with Unipotent Part

Definition 1.20. A formal k-group $\mathcal{F}$ is called *dual-algebraic* if its Cartier-dual $\mathcal{F}^\vee$ is algebraic. The category of dual-algebraic formal k-groups is denoted by $d\mathcal{G}f/k$.

Proposition 1.21. *A formal k-group $\mathcal{F}$ is dual-algebraic if and only if the following conditions are satisfied:*

(1) $\mathcal{F}(\overline{k})$ *is of finite type,*

(2) *for* $\mathrm{char}(k) = 0$*:* $\mathrm{Lie}\,(\mathcal{F})$ *is finite dimensional,*
for $\mathrm{char}(k) > 0$*:* $\mathcal{F}_{\mathrm{inf}}$ *is a quotient of* $\big({}_r\widehat{\mathbb{W}}\big)^n$ *for some* $r, n \in \mathbb{N}$
(see Proposition 1.18 for the definition of ${}_r\widehat{\mathbb{W}}$*).*

Proof. The splitting $\mathcal{F}^\vee = \mathcal{F}^\vee_{\text{ét}} \times \mathcal{F}^\vee_{\mathrm{inf}}$ gives the decomposition of the affine group $\mathcal{F}^\vee$ into multiplicative part and unipotent part, according to Propositions 1.15 and 1.17. Then that statement follows directly from Propositions 1.16 and 1.18 for $\mathrm{char}(k) > 0$. For $\mathrm{char}(k) = 0$, the assertion in (2) is due to the fact that the Lie functor yields an equivalence between the category of commutative infinitesimal formal k-groups and the category of k-vector spaces, see [SGA3, $\mathrm{VII_B}$, 3.2.2]. ■

Definition 1.22. A *1-motive with unipotent part* is a tuple $M = (\mathcal{F}, L, A, G, \mu)$, where

(a) $\mathcal{F}$ is a dual-algebraic formal group (Definition 1.20),
(b) L is an affine algebraic group,
(c) A is an abelian variety,
(d) G is an extension of A by L,
(e) $\mu : \mathcal{F} \longrightarrow G$ is a homomorphism in $\mathcal{A}b/k$.

A *homomorphism between 1-motives with unipotent part* $M = (\mathcal{F}, L, A, G, \mu)$ and $N = (\mathcal{E}, \Lambda, B, H, \nu)$ is a tuple $h = (\varphi, \lambda, \alpha, \gamma)$ of homomorphisms $\varphi : \mathcal{E} \to \mathcal{F}, \lambda : L \to \Lambda, \alpha : A \to B, \gamma : G \to H$, compatible with the structures of M and N as 1-motives with unipotent part, i.e. giving an obvious commutative diagram.

For convenience, we will refer to a *1-motive with unipotent part* only as a *1-motive*.

If G is a smooth connected algebraic group, it admits a canonical decomposition $0 \to L \to G \to A \to 0$ as an extension of an abelian variety A by a connected affine algebraic group L, according to Theorem 1.5 of Chevalley. Thus a homomorphism $\mu : \mathcal{F} \longrightarrow G$ in $\mathcal{A}b/k$ gives rise to a 1-motive $M = (\mathcal{F}, L, A, G, \mu)$ that we will denote just by $M = \left[\mathcal{F} \xrightarrow{\mu} G\right]$.

1.2.2 Duality of 1-Motives

Let L be an affine algebraic group and A an abelian variety. The Cartier-dual of L is given by $L^\vee = \mathrm{Hom}_{\mathcal{A}b/k}(L, \mathbb{L}_?)$ (Lemma 1.14), the completion of the dual abelian variety of A by the formal k-group functor $\widehat{A^\vee} = \mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_?)$ (Proposition 1.19). Since $\widehat{A^\vee}$ is a formal group, there exists an affine group $\mathfrak{L}$ such that $\widehat{A^\vee} \cong \underline{\mathrm{Hom}}_{\mathcal{A}b/k}(\mathfrak{L}, \mathbb{G}_\mathrm{m}) = \mathrm{Hom}_{\mathcal{A}b/k}(\mathfrak{L}, \mathbb{L}_?)$. Then the formal k-group functors $L^\vee$ resp. $\widehat{A^\vee}$ extend to functors from affine groups $\mathcal{G}a/k$ to abelian groups $\mathbf{Ab}$, defined by $L^\vee(\Lambda) = \mathrm{Hom}_{\mathcal{A}b/k}(L, \Lambda)$ resp. $\widehat{A^\vee}(\Lambda) = \mathrm{Hom}_{\mathcal{A}b/k}(\mathfrak{L}, \Lambda)$ for $\Lambda \in \mathcal{G}a/k$. Using the Yoneda Lemma, one can show that any transformation $\vartheta \in \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right)$ becomes a homomorphism in the category $\mathbf{Fctr}(\mathcal{G}a/k, \mathbf{Ab})$. As natural transformations commute with functoriality maps, we obtain in particular

Lemma 1.23. *A homomorphism* $\vartheta \in \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right)$ *commutes with homomorphisms in* $\mathcal{G}a/k$ *as follows:*

$$\vartheta\left(\mu_* \nu\right) = \mu_* \vartheta\left(\nu\right)$$

for $\nu \in L^\vee(S) = \mathrm{Hom}_{\mathcal{A}b/k}(L, \mathbb{L}_S)$ *and* $\mu \in \mathbb{L}_S^\vee(R) = \mathrm{Hom}_{\mathcal{A}b/k}(\mathbb{L}_S, \mathbb{L}_R)$, *where* $S, R \in \mathbf{Art}/k$. *(Here* $\vartheta(\nu) \in \mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_S)$, $\vartheta(\mu_*\nu) \in \mathrm{Ext}_{\mathcal{A}b/k}(A, \mathbb{L}_R)$ *and* $\mu_*\vartheta(\nu)$ *is the push-out of* $\vartheta(\nu)$ *via* μ*.)*

Theorem 1.24. *Let* L *be an affine algebraic group and* A *an abelian variety. There is a canonical isomorphism of abelian groups*

$$\Phi : \mathrm{Ext}_{\mathcal{A}b/k}\left(A, L\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right).$$

Proof. In a first step, we will show the statement for $L = \mathbb{L}_S$, where $S \in \mathbf{Art}/k$. In a second step, we will derive the result for arbitrary $L \in a\mathcal{G}a/k$ by choosing an embedding $L \subset \mathbb{L}_S$ and diagram chasing.

Step 1: Assume $L = \mathbb{L}_S$ for some $S \in \mathbf{Art}/k$.
Construct a map $\Phi : \mathrm{Ext}_{\mathcal{A}b/k}\left(A, L\right) \longrightarrow \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right)$ as follows: Let G be an extension of A by L. According to Lemma 1.14 the R-valued points of $L^\vee$ are $L^\vee(R) = \mathrm{Hom}_{\mathcal{A}b/k}\left(L, \mathbb{L}_R\right)$, i.e. $\lambda \in L^\vee(R)$ gives rise to a homomorphism $L \longrightarrow \mathbb{L}_R$. Then the image of λ under $\Phi(G)$ is the push-out $\lambda_* G \in \mathrm{Ext}_{\mathcal{A}b/k}\left(A, \mathbb{L}_R\right) = A^\vee(R)$, cf. Lemma 1.19.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\lambda"'] & G \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{L}_R \arrow[r] & \lambda_* G \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Conversely, a map $\Psi : \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right) \longrightarrow \mathrm{Ext}_{\mathcal{A}b/k}\left(A, L\right)$ is obtained as follows: Given a homomorphism $\vartheta : L^\vee \longrightarrow A^\vee$, then define $\Psi(\vartheta)$ to be the image of $\mathrm{id}_L \in \mathrm{Hom}_{\mathcal{A}b/k}(L, L) = L^\vee(S)$ in $\mathrm{Ext}_{\mathcal{A}b/k}(A, L) = A^\vee(S)$ under $\vartheta\,(S)$. By construction, Φ and Ψ are inverse to each other, cf. Lemma 1.23. Moreover, one checks that Φ and Ψ are homomorphisms of abelian groups.

Step 2: Let L be an arbitrary affine algebraic group.
By Lemma 1.12 there is a finite k-algebra S and a monomorphism of affine groups $\iota : L \longrightarrow \mathbb{L}_S$. Since $a\mathcal{G}a/k$ is an abelian category, the quotient $\mathbb{L}_S/L$ is again an affine algebraic group. By Cartier duality, we have exact sequences
$0 \to L \to \mathbb{L}_S \to \mathbb{L}_S/L \to 0$ and $0 \leftarrow L^\vee \leftarrow (\mathbb{L}_S)^\vee \leftarrow (\mathbb{L}_S/L)^\vee \leftarrow 0$.
The functor $\mathrm{Hom}_{\mathcal{A}b/k}(?, A^\vee)$ is left-exact and $\mathrm{Ext}_{\mathcal{A}b/k}(A, ?)$ is left-exact on $\mathcal{G}a/k$, since $\mathrm{Hom}_{\mathcal{A}b/k}(A, L) = 0$ for any affine group L. The homomorphism $\Phi : \mathrm{Ext}_{\mathcal{A}b/k}\left(A, L\right) \longrightarrow \mathrm{Hom}_{\mathcal{A}b/k}\left(L^\vee, A^\vee\right)$ from Step 1 is defined for arbitrary

$L \in a\mathcal{G}a/k$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(A, L) \arrow[r] \arrow[d, "\Phi_L"'] & \mathrm{Ext}(A, \mathbb{L}_S) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}(A, \mathbb{L}_S/L) \arrow[d, "\Phi_{\mathbb{L}_S/L}"] \\
0 \arrow[r] & \mathrm{Hom}(L^\vee, A^\vee) \arrow[r] & \mathrm{Hom}\big((\mathbb{L}_S)^\vee, A^\vee\big) \arrow[r] & \mathrm{Hom}\big((\mathbb{L}_S/L)^\vee, A^\vee\big).
\end{tikzcd}$$

Here $\mathrm{Hom}_{\mathcal{A}b/k}(\iota^\vee, A^\vee) \circ \Phi_L \cong \mathrm{Ext}_{\mathcal{A}b/k}(A, \iota)$ is injective, hence Φ_L is injective. The same procedure for $\mathbb{L}_S/L$ instead of L yields the injectivity of $\Phi_{\mathbb{L}_S/L}$. Now a refined version of the Five Lemma (see [KwS, Lemma 8.3.13 (ii)]), which is actually a Four Lemma, implies that Φ_L is surjective. Thus Φ_L is an isomorphism. ■

Remark 1.25. The isomorphism Φ in Theorem 1.24 sends $G \in \mathrm{Ext}(A, L)$ to the connecting homomorphism $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(L, \mathbb{G}_\mathrm{m}) \longrightarrow \underline{\mathrm{Ext}}_{\mathcal{A}b/k}(A, \mathbb{G}_\mathrm{m})$ in the long exact cohomology sequence obtained from applying $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}(?, \mathbb{G}_\mathrm{m})$ to the short exact sequence $0 \to L \to G \to A \to 0$.

Definition 1.26. Let A be an abelian variety, L an affine algebraic group and $G \in \mathrm{Ext}_{\mathcal{A}b/k}(A, L)$. The dual of the 1-motive $\big[0 \to G\big] := \big(0, L, A, G, 0\big)$ is by definition the 1-motive $\left[L^\vee \overset{\Phi(G)}{\longrightarrow} A^\vee\right] := \big(L^\vee, 0, A^\vee, A^\vee, \Phi(G)\big)$.

Let A be an abelian variety, $\mathcal{F}$ a dual-algebraic formal group and $\mu \in \mathrm{Hom}_{\mathcal{A}b/k}(\mathcal{F}, A)$. Then the dual of the 1-motive $\left[\mathcal{F} \overset{\mu}{\longrightarrow} A\right] := \big(\mathcal{F}, 0, A, A, \mu\big)$ is by definition the 1-motive $\big[0 \longrightarrow \Phi^{-1}(\mu)\big] := \big(0, \mathcal{F}^\vee, A^\vee, \Phi^{-1}(\mu), 0\big)$. (Here Φ is the isomorphism from Theorem 1.24.)

Notation 1.27. We denote the dual of a 1-motive M by $M^\vee$.

Remark 1.28. It is clear from construction that the double dual $M^{\vee\vee}$ of a pure 1-motive M is canonically isomorphic to M.

Remark 1.29. Duality of pure 1-motives with unipotent part extends to arbitrary 1-motives with unipotent part as follows: The dual of a 1-motive $M = (\mathcal{F}, L, A, G, \mu)$ is the 1-motive $M^\vee = (L^\vee, \mathcal{F}^\vee, A^\vee, G^\alpha, \mu^\vee)$, where $G^\alpha = \underline{\mathrm{Ext}}_{\mathcal{C}(\mathcal{A}b/k)}\big([\mathcal{F} \to A], \mathbb{G}_\mathrm{m}\big)$ and $\mu^\vee$ is the connecting homomorphism $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\big(L, \mathbb{G}_\mathrm{m}\big) \longrightarrow \underline{\mathrm{Ext}}_{\mathcal{C}(\mathcal{A}b/k)}\big([\mathcal{F} \to A], \mathbb{G}_\mathrm{m}\big)$ in the long exact cohomology sequence associated to $0 \longrightarrow [0 \to L] \longrightarrow [\mathcal{F} \to G] \longrightarrow [\mathcal{F} \to A] \longrightarrow 0$. Here $\mathcal{C}(\mathcal{A}b/k)$ is the category of complexes of sheaves of abelian groups, and in $[\mathcal{F} \to A]$, $\mathcal{F}$ is placed in degree -1 and A in degree 0. (In this note we will only use duality of 1-motives of pure type as in Definition 1.26.)

Proposition 1.30. *Duality of 1-motives is functorial, i.e. duality assigns to a homomorphism of 1-motives $h : M \longrightarrow N$ a dual homomorphism $h^\vee : N^\vee \longrightarrow M^\vee$.*

Proof. Functoriality comes from the fact that duality of 1-motives is derived from the functor $\underline{\mathrm{Hom}}_{\mathcal{C}(\mathcal{A}b/k)}(?, \mathbb{G}_{\mathrm{m}})$, cf. Remark 1.25. ■

2 Universal Rational Maps

Let X be a variety over a field k. The classical Albanese variety $\mathrm{Alb}(X)$ of X (as in [Lang, II, § 3]) is an abelian variety, defined together with the Albanese map $\mathrm{alb} : X \dashrightarrow \mathrm{Alb}(X)$ by the following universal mapping property: for every rational map $\varphi : X \dashrightarrow A$ to an abelian variety A there is a unique homomorphism $h : \mathrm{Alb}(X) \longrightarrow A$ such that $\varphi = h \circ \mathrm{alb}$ up to translation by a constant $a \in A(k)$. Now we replace in this definition the category of abelian varieties by a subcategory $\mathfrak{C}$ of the category of commutative algebraic groups. A result of Serre [Ser1, No. 6, Théorème 8, p. 10-14] says that if the category $\mathfrak{C}$ contains the additive group $\mathbb{G}_{\mathrm{a}}$ and X is a variety of dimension > 0, there does not exist an Albanese variety in $\mathfrak{C}$ that is universal for all rational maps from X to algebraic groups in $\mathfrak{C}$. One is therefore led to restrict the class of considered rational maps. This motivates the concept of *categories of rational maps from X to commutative algebraic groups* (Definition 2.7), and to ask for the existence of universal objects for such categories.

For k an algebraically closed field with $\mathrm{char}(k) = 0$, in [Ru, Section 2] a criterion is given, for which categories **Mr** of rational maps from a smooth proper variety X over k to algebraic groups there exists a universal object $\mathrm{Alb}_{\mathbf{Mr}}(X)$, as well as an explicit construction of these universal objects via duality of 1-motives. Similar results are true for perfect base field of arbitrary characteristic as well, as we will see in this section.

2.1 Relative Cartier Divisors

The construction of such universal objects as above involves the functor $\underline{\mathrm{Div}}_X : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ of families of Cartier divisors, given by

$$\underline{\mathrm{Div}}_X(R) = \left\{ \begin{array}{c} \text{Cartier divisors } \mathcal{D} \text{ on } X \times_k \operatorname{Spec} R \\ \text{whose fibres } \mathcal{D}_p \text{ define Cartier divisors on } X \times_k \{p\} \\ \text{for all } \ p \in \operatorname{Spec} R \end{array} \right\}$$

for each k-algebra R, and for a homomorphism $h : R \longrightarrow S$ in $\mathbf{Alg}/k$ the induced homomorphism $\underline{\mathrm{Div}}_X(h) : \underline{\mathrm{Div}}_X(R) \longrightarrow \underline{\mathrm{Div}}_X(S)$ in $\mathbf{Ab}$ is the pull-back of Cartier divisors on $X \times_k \operatorname{Spec} R$ to those on $X \times_k \operatorname{Spec} S$. The

elements of $\underline{\mathrm{Div}}_X(R)$ are called *relative Cartier divisors*. See [Ru, No. 2.1] for more details on $\underline{\mathrm{Div}}_X$.

We will be mainly concerned with the completion $\widehat{\underline{\mathrm{Div}}_X} : \mathbf{Art}/k \longrightarrow \mathbf{Ab}$ of $\underline{\mathrm{Div}}_X$, which is given for every finite k-algebra R by

$$\widehat{\underline{\mathrm{Div}}_X}(R) = \Gamma\big(X \otimes R, (\mathcal{K}_X \otimes_k R)^* \,/\, (\mathcal{O}_X \otimes_k R)^*\,\big).$$

We will regard $\widehat{\underline{\mathrm{Div}}_X}$ as a subsheaf of $\underline{\mathrm{Div}}_X$, cf. Remark 1.8.

Proposition 2.1. $\widehat{\underline{\mathrm{Div}}_X}$ *is a formal k-group.*

Proof. According to Proposition 1.7 it suffices to show that $\widehat{\underline{\mathrm{Div}}_X}$ is left-exact. We are going to show that $\widehat{\underline{\mathrm{Div}}_X}$ is the composition of left-exact functors.

Let R be a finite k-algebra. $\widehat{\underline{\mathrm{Div}}_X}(R) = \Gamma\big(X, \mathcal{Q}(R)\big)$ is the abelian group of global sections of the sheaf $\mathcal{Q}(R) := (\mathrm{pr}_X)_*\big(\,(\mathcal{K}_X \otimes R)^* / (\mathcal{O}_X \otimes R)^*\,\big)$, where $\mathrm{pr}_X : X \otimes R \longrightarrow X$ is the projection. The global section functor $\Gamma(X, ?)$ is known to be left-exact. We show that the formal k-group functor $\mathcal{Q} : \mathbf{Art}/k \longrightarrow \mathcal{A}b/X$ (with values in the category of sheaves of abelian groups over X) commutes with finite projective limits (hence is left-exact):

Let (R_i) be a projective system of local finite k-algebras, with homomorphisms $h_{ij} : R_j \longrightarrow R_i$ for $i < j$. We have projections $\mathrm{pr}_j : \varprojlim R_i \longrightarrow R_j$ for each j, which commute with the h_{ij}. Functoriality of $\mathcal{Q}$ in $R \in \mathbf{Art}/k$ induces homomorphisms $\mathcal{Q}(h_{ij}) : \mathcal{Q}(R_j) \longrightarrow \mathcal{Q}(R_i)$ and $\mathcal{Q}(\mathrm{pr}_j) : \mathcal{Q}(\varprojlim R_i) \longrightarrow \mathcal{Q}(R_j)$, which commute. The universal property of $\varprojlim \mathcal{Q}(R_i)$ yields a unique homomorphism of sheaves $\mathcal{Q}(\varprojlim R_i) \longrightarrow \varprojlim \mathcal{Q}(R_i)$. A homomorphism of sheaves is an isomorphism if and only if it is an isomorphism on stalks. Therefore it remains to show that the stalks $\mathcal{Q}_q : \mathbf{Art}/k \longrightarrow \mathbf{Ab}$ for $q \in X$ are left-exact in $R \in \mathbf{Art}/$. We have $\mathcal{Q}_q = \mathbb{G}_{\mathrm{m}}(\mathcal{K}_{X,q} \otimes_k ?) / \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{X,q} \otimes_k ?)$. The tensor product over a field $\mathcal{A} \otimes_k ? : \mathbf{Art}/k \longrightarrow \mathbf{Alg}/k$ is exact for any k-algebra $\mathcal{A}$. Also the sheaf $\mathbb{G}_{\mathrm{m}} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ is left-exact. Therefore the formal k-group functors $\mathbb{G}_{\mathrm{m}}(\mathcal{K}_{X,q} \otimes_k ?)$ and $\mathbb{G}_{\mathrm{m}}(\mathcal{O}_{X,q} \otimes_k ?)$ are formal k-groups. Since the category $\mathcal{G}f/k$ of formal k-groups is abelian, the quotient $\mathcal{Q}_q$ of these two formal k-groups is again a formal k-group. ■

Definition 2.2. Let R be a finite k-algebra. If $D \in \big(\widehat{\underline{\mathrm{Div}}_X}\big)_{\text{ét}}(R)$, then $\mathrm{Supp}(D)$ denotes the locus of zeroes and poles of local sections $(f_\alpha)_\alpha$ of $(\mathcal{K}_X \otimes R_{\mathrm{red}})^*$ representing $D \in \Gamma\big(\,(\mathcal{K}_X \otimes R_{\mathrm{red}})^* / (\mathcal{O}_X \otimes R_{\mathrm{red}})^*\,\big)$.
If $\delta \in \big(\widehat{\underline{\mathrm{Div}}_X}\big)_{\mathrm{inf}}(R)$, then $\mathrm{Supp}(\delta)$ denotes the locus of poles of local sections

$(g_\alpha)_\alpha$ of $\mathbb{U}_R(\mathcal{K}_X)$ representing $\delta \in \Gamma\big(\, \mathbb{U}_R(\mathcal{K}_X)/\, \mathbb{U}_R(\mathcal{O}_X)\big) = \big(\widehat{\underline{\mathrm{Div}}_X}\big)_{\mathrm{inf}}(R)$. (The functor $\mathbb{U}_R$ is defined in 1.1.6.)

Definition 2.3. Let $\mathcal{F}$ be a formal subgroup of $\underline{\mathrm{Div}}_X$. The support of $\mathcal{F}$ is defined to be

$$\mathrm{Supp}(\mathcal{F}) = \bigcup_{\substack{R\in\mathbf{Art}/k \\ \mathcal{D}\in\mathcal{F}(R)}} \mathrm{Supp}(\mathcal{D})$$

where we use the decomposition $\mathcal{F} = \mathcal{F}_{\text{ét}} \times \mathcal{F}_{\mathrm{inf}}$ and Definition 2.2.

Suppose now that X is a geometrically irreducible smooth proper variety over a perfect field k. Then the Picard functor $\underline{\mathrm{Pic}}_X$ is represented by a separated algebraic space Pic_X, whose identity component Pic_X^0 is a proper scheme over k (see [BLR, No. 8.4, Thm. 3]). The underlying reduced scheme $\mathrm{Pic}_X^{0,\mathrm{red}}$ of Pic_X^0 is an abelian variety, called the *Picard variety* of X. The subfunctor of $\underline{\mathrm{Pic}}_X$ that is represented by $\mathrm{Pic}_X^{0,\mathrm{red}}$ will be denoted by $\underline{\mathrm{Pic}}_X^{0,\mathrm{red}}$.

There is a natural transformation

$$\mathrm{cl} : \underline{\mathrm{Div}}_X \longrightarrow \underline{\mathrm{Pic}}_X \,.$$

We define $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ to be the subfunctor of $\underline{\mathrm{Div}}_X$ given by

$$\underline{\mathrm{Div}}_X^{0,\mathrm{red}} = \underline{\mathrm{Div}}_X \times_{\underline{\mathrm{Pic}}_X} \underline{\mathrm{Pic}}_X^{0,\mathrm{red}} \,.$$

2.2 Categories of Rational Maps to Algebraic Groups

Let X be a smooth proper variety over an algebraically closed field k of arbitrary characteristic. All considered algebraic groups and formal groups are commutative by definition (No.s 1.1.3 and 1.1.4).

2.2.1 Induced Transformation

Let G be a smooth connected algebraic group, and let $0 \to L \to G \to A \to 0$ be the canonical decomposition of G, where A is an abelian variety and L an affine smooth connected algebraic group (Theorem 1.5). Write $L = T \times_k U$ where T is a torus and U is unipotent (Theorem 1.4). Since k is algebraically closed, $T \cong (\mathbb{G}_\mathrm{m})^t$ for some $t \in \mathbb{N}$. If k is of characteristic 0, one has $U \cong (\mathbb{G}_\mathrm{a})^s$ for some $s \in \mathbb{N}$ ([DG, IV, § 2, 4.2]). If k is of characteristic $p > 0$, the unipotent group U is embedded into a finite direct sum $(\mathbb{W}_r)^s$ of Witt vector groups for some $r, s \in \mathbb{N}$ ([DG, V, § 1, 2.5]).

Since $\mathrm{H}^1_{\mathrm{fppf}}\big(\operatorname{Spec}(\mathcal{O}_{X,q}), \mathbb{G}_\mathrm{m}\big) = 0$ and $\mathrm{H}^1_{\mathrm{fppf}}\big(\operatorname{Spec}(\mathcal{O}_{X,q}), U\big) = 0$ for any point q of X, we have exact sequences

$$0 \longrightarrow L\left(\mathcal{K}_{X,q}\right) \longrightarrow G\left(\mathcal{K}_{X,q}\right) \longrightarrow A\left(\mathcal{K}_{X,q}\right) \longrightarrow 0$$

$$0 \longrightarrow L\left(\mathcal{O}_{X,q}\right) \longrightarrow G\left(\mathcal{O}_{X,q}\right) \longrightarrow A\left(\mathcal{O}_{X,q}\right) \longrightarrow 0.$$

Since a rational map to an abelian variety is defined at every smooth point (see [Lang, II, § 1, Thm. 2]), we have $A\left(\mathcal{K}_{X,q}\right) = A\left(\mathcal{O}_{X,q}\right)$ for every point q of X. Hence the canonical map

$$L\left(\mathcal{K}_{X,q}\right)/L\left(\mathcal{O}_{X,q}\right) \longrightarrow G\left(\mathcal{K}_{X,q}\right)/G\left(\mathcal{O}_{X,q}\right)$$

is bijective. By Cartier-duality, we have a pairing

$$\langle ?, ? \rangle : L^\vee \times \Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(\mathbb{G}_\mathrm{m}(\mathcal{K}_X \otimes \_)/\mathbb{G}_\mathrm{m}(\mathcal{O}_X \otimes \_)\big).$$

Definition 2.4. Let $\varphi : X \dashrightarrow G$ be a rational map to a smooth connected algebraic group G, let L be the affine part of G. Then $\tau_\varphi : L^\vee \longrightarrow \widehat{\underline{\mathrm{Div}}_X}$ denotes the induced transformation given by $\langle ?, \ell_\varphi \rangle$, where ℓ_φ is the image of $\varphi \in G\left(\mathcal{K}_X\right)$ in $\Gamma\big(G(\mathcal{K}_X)/G(\mathcal{O}_X)\big) \xrightarrow{\sim} \Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big)$. By construction, τ_φ is a homomorphism of formal k-group functors.

Lemma 2.5. *Let G be a smooth connected algebraic group, let L be the affine part of G. Let $\varphi : X \dashrightarrow G$ be a rational map. Let $\tau_\varphi : L^\vee \longrightarrow \widehat{\underline{\mathrm{Div}}_X}$ be the induced transformation. Then $\operatorname{im}(\tau_\varphi)$ is a dual-algebraic formal group.*

Proof. $\widehat{\underline{\mathrm{Div}}_X}$ is a formal group by Proposition 2.1, and $\mathcal{G}f/k$ is a full subcategory of $\mathbf{Fctr}(\mathbf{Art}/k, \mathbf{Ab})$. Therefore $\tau_\varphi : L^\vee \longrightarrow \widehat{\underline{\mathrm{Div}}_X}$ is a homomorphism of formal groups. Since $\mathcal{G}f/k$ is an abelian category, kernel and image of the homomorphism τ_φ are formal groups. Since L is algebraic, $L^\vee$ is dual-algebraic and hence $\operatorname{im}(\tau_\varphi)$, as a quotient of $L^\vee$, is dual-algebraic. ∎

Lemma 2.6. *Let $G \in \operatorname{Ext}(A, L)$ be a smooth connected algebraic group. Let $\varphi : X \dashrightarrow G$ a rational map. Let $\tau_\varphi : L^\vee \longrightarrow \widehat{\underline{\mathrm{Div}}_X}$ be the induced transformation. Then $\operatorname{im}(\tau_\varphi)$ is contained in the completion of $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$.*

Proof. As A is an abelian variety, the composition $X \overset{\varphi}{\dashrightarrow} G \xrightarrow{\rho} A$ extends to a morphism $\overline{\varphi} : X \longrightarrow A$. The description of the induced transformation τ_φ in terms of local sections into principal fibre bundles as given in [Ru, No. 2.2] shows: the composition

$$L^\vee \xrightarrow{\tau_\varphi} \underline{\mathrm{Div}}_X \xrightarrow{\mathrm{cl}} \underline{\mathrm{Pic}}_X$$

is given by $\lambda \longmapsto \lambda_* G_X$, where $\lambda_* G$ is the push-out of $G \in \mathrm{Ext}_{\mathcal{A}b/k}(A, L)$ via $\lambda \in L^\vee(R) = \mathrm{Hom}_{\mathcal{A}b/k}(L, \mathbb{L}_R)$, and $G_X = G \times_A X$ is the fibre-product of G and X over A. Hence it comes down to show that for each $R \in \mathbf{Art}/k$, each $\lambda \in L^\vee(R)$ the $\mathbb{L}_R$-bundle $\lambda_* G_X$ yields an element of $\mathrm{Pic}_X^{0,\mathrm{red}}(R)$.

The universal mapping property of the classical Albanese $\mathrm{Alb}(X)$ yields that $\overline{\varphi}$ factors through $\mathrm{Alb}(X)$. Hence the pull-back $G_X = G \times_A X$ over X is a pull-back of $G_{\mathrm{Alb}} = G \times_A \mathrm{Alb}(X)$ over $\mathrm{Alb}(X)$.

$$\begin{tikzcd}
G_X \arrow[r] \arrow[d] & G_{\mathrm{Alb}} \arrow[r] \arrow[d] & G \arrow[d, "\rho"] \\
X \arrow[r] \arrow[u, bend right, "\varphi_X"'] \arrow[ur, bend left, "\varphi_{\mathrm{Alb}}"] \arrow[urr, bend right=10, "\varphi"] & \mathrm{Alb}(X) \arrow[r] & A
\end{tikzcd}$$

Then for each $\lambda \in L^\vee(R)$ the $\mathbb{L}_R$-bundle $\lambda_* G_{\mathrm{Alb}}$ over $\mathrm{Alb}(X)$ is an element of $\mathrm{Ext}_{\mathcal{A}b/k}(\mathrm{Alb}(X), \mathbb{L}_R)$, hence gives an element of $\mathrm{Pic}^0_{\mathrm{Alb}(X)}(R)$. Since $\mathrm{Alb}(X) = \left(\mathrm{Pic}_X^{0,\mathrm{red}}\right)^\vee$ is the dual abelian variety of $\mathrm{Pic}_X^{0,\mathrm{red}}$, we have an isomorphism $\mathrm{Pic}^0_{\mathrm{Alb}(X)} \xrightarrow{\sim} \mathrm{Pic}_X^{0,\mathrm{red}}$, $P \longmapsto P_X = P \times_{\mathrm{Alb}(X)} X$. As $\lambda_* G_X = \lambda_* G_{\mathrm{Alb}} \times_{\mathrm{Alb}(X)} X$, it holds $\lambda_* G_X \in \mathrm{Pic}_X^{0,\mathrm{red}}(R)$. ■

2.2.2 Definition of a Category of Rational Maps

Definition 2.7. A *category* **Mr** *of rational maps from* X *to algebraic groups* is a category satisfying the following conditions: The objects of **Mr** are rational maps $\varphi : X \dashrightarrow G$, where G is a smooth connected algebraic group. The morphisms of **Mr** between two objects $\varphi : X \dashrightarrow G$ and $\psi : X \dashrightarrow H$ are given by the set of those affine homomorphisms (homomorphisms of algebraic groups composed with a translation) $h : G \longrightarrow H$ such that $h \circ \varphi = \psi$.

Remark 2.8. Let $\varphi : X \dashrightarrow G$ and $\psi : X \dashrightarrow H$ be two rational maps from X to algebraic groups. Then Definition 2.7 implies that for any category **Mr** of rational maps from X to algebraic groups containing φ and ψ as objects the set of morphisms $\mathrm{Hom}_{\mathbf{Mr}}(\varphi, \psi)$ is the same. Therefore two categories **Mr** and **Mr**$'$ of rational maps from X to algebraic groups are equivalent if every object of **Mr** is isomorphic to an object of **Mr**$'$.

Definition 2.9. The category of rational maps from X to abelian varieties is denoted by **Mav**.

Definition 2.10. Let $\mathcal{F}$ be a dual-algebraic formal subgroup of $\underline{\mathrm{Div}}_X$. Then $\mathbf{Mr}_{\mathcal{F}}$ denotes the category of all those rational maps $\varphi : X \dashrightarrow G$ from X to algebraic groups for which the image of the induced transformation $\tau_\varphi : L^\vee \longrightarrow \underline{\mathrm{Div}}_X$ (Definition 2.4) lies in $\mathcal{F}$, i.e. which induce a homomorphism of formal groups $L^\vee \longrightarrow \mathcal{F}$, where L is the affine part of G.

$$\mathbf{Mr}_{\mathcal{F}} = \{\varphi : X \dashrightarrow G \mid \operatorname{im} \tau_\varphi \subset \mathcal{F}\}$$

2.3 Universal Objects

Let X be a smooth proper variety over k (an algebraically closed field of arbitrary characteristic). Algebraic groups are always assumed to be smooth and connected, unless stated otherwise.

2.3.1 Existence and Construction

Definition 2.11. Let $\mathbf{Mr}$ be a category of rational maps from X to algebraic groups. Then $(u : X \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$ is called a *universal object for* $\mathbf{Mr}$ if it admits the universal mapping property in $\mathbf{Mr}$: For all $(\varphi : X \dashrightarrow G) \in \mathbf{Mr}$ there is a unique affine homomorphism $h : \mathcal{U} \longrightarrow G$ such that $\varphi = h \circ u$.

For the category $\mathbf{Mav}$ of morphisms from X to abelian varieties (Definition 2.9) there exists a universal object, the *Albanese mapping* to the *Albanese variety,* denoted by $\mathrm{alb} : X \longrightarrow \mathrm{Alb}(X)$. This is a classical result (see [Lang], [Msa], [Ser1]). The Albanese variety $\mathrm{Alb}(X)$ is an abelian variety, dual to the Picard variety $\mathrm{Pic}_X^{0,\mathrm{red}}$.

In the following we consider categories $\mathbf{Mr}$ of rational maps from X to algebraic groups satisfying the following conditions:

($\diamondsuit$ 1) $\mathbf{Mr}$ contains the category $\mathbf{Mav}$.

($\diamondsuit$ 2) If $(\varphi : X \dashrightarrow G) \in \mathbf{Mr}$ and $h : G \longrightarrow H$ is an affine homomorphism of smooth connected algebraic groups, then $h \circ \varphi \in \mathbf{Mr}$.

Theorem 2.12. *Let* $\mathbf{Mr}$ *be a category of rational maps from* X *to algebraic groups satisfying* ($\diamondsuit$ 1, 2)*. Then for* $\mathbf{Mr}$ *there exists a universal object* $(u : X \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$ *if and only if there is a dual-algebraic formal subgroup* $\mathcal{F}$ *of* $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ *such that* $\mathbf{Mr}$ *is equivalent to* $\mathbf{Mr}_{\mathcal{F}}$ *(where* $\mathbf{Mr}_{\mathcal{F}}$ *is the category of rational maps which induce a homomorphism of formal groups to* $\mathcal{F}$*, see Definition 2.10).*

Proof. ($\Longleftarrow$) Assume that $\mathbf{Mr}$ is equivalent to $\mathbf{Mr}_{\mathcal{F}}$, where $\mathcal{F}$ is a dual-algebraic formal group in $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. The first step is the construction of an

algebraic group $\mathcal{U}$ and a rational map $u : X \dashrightarrow \mathcal{U}$. In a second step the universality of $u : X \dashrightarrow \mathcal{U}$ for $\mathbf{Mr}_{\mathcal{F}}$ will be shown.

Step 1: Construction of $u : X \dashrightarrow \mathcal{U}$.
X is a smooth proper variety over k, thus the functor $\underline{\mathrm{Pic}}^0_X$ is represented by an algebraic group Pic^0_X whose underlying reduced scheme $\mathrm{Pic}^{0,\mathrm{red}}_X$, the Picard variety of X, is an abelian variety. The class map $\underline{\mathrm{Div}}_X \longrightarrow \underline{\mathrm{Pic}}_X$ induces a homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^{0,\mathrm{red}}_X$.

We obtain a 1-motive $M = \big[\mathcal{F} \longrightarrow \mathrm{Pic}^{0,\mathrm{red}}_X\big]$. Since $\mathrm{Pic}^{0,\mathrm{red}}_X$ is an abelian variety, the dual 1-motive of M is of the form $M^\vee = [0 \to G]$, where G is a smooth connected algebraic group. Then define $\mathcal{U}$ to be this algebraic group. The canonical decomposition $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ is the extension of $\mathcal{A} = \mathrm{Alb}(X) = \big(\mathrm{Pic}^{0,\mathrm{red}}_X\big)^\vee$ by $\mathcal{L} = \mathcal{F}^\vee$ induced by the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^{0,\mathrm{red}}_X$ (Theorem 1.24).

By Lemma 1.12 there exists a finite k-algebra S and an injective homomorphism of affine algebraic groups $\lambda : \mathcal{L} \longrightarrow \mathbb{L}_S$. Assume first that $\mathcal{L} = \mathbb{L}_S$. The homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^{0,\mathrm{red}}_X$ on S-valued points $\mathbb{L}^\vee_S(S) \longrightarrow \mathcal{A}^\vee(S)$ has values in $\mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathbb{L}_S)$, according to Proposition 1.19. The proof of Theorem 1.24 shows that $\mathcal{U} \in \mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathbb{L}_S)$ is the image of $\iota := \mathrm{id}_{\mathbb{L}_S} \in \mathrm{Hom}_{\mathcal{A}b/k}(\mathbb{L}_S, \mathbb{L}_S) = \mathbb{L}^\vee_S(S)$. The case $\mathcal{L} \subset \mathbb{L}_S$ is achieved by a descent argument. The composition of the pull-back $\lambda^\vee : \mathbb{L}^\vee_S \longrightarrow \mathcal{L}^\vee$ with $\mathcal{L}^\vee = \mathcal{F} \longrightarrow \mathrm{Pic}^{0,\mathrm{red}}_X = \mathcal{A}^\vee$ yields a homomorphism $\phi : \mathbb{L}^\vee_S \longrightarrow \mathcal{A}^\vee$. As ϕ factors through $\mathcal{L}^\vee$, the image $\mathcal{U} = \phi(\iota)$ of $\iota \in \mathbb{L}^\vee_S(S)$ lies actually in $\mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathcal{L})$, cf. Step 2 in the proof of Theorem 1.24. Define the rational map $u : X \dashrightarrow \mathcal{U}$ by the condition that the induced transformation $\tau_u : \mathcal{F} \longrightarrow \underline{\mathrm{Div}}^{0,\mathrm{red}}_X$ from Definition 2.4 is the inclusion. This yields $\ell_u \in \Gamma\big(\mathcal{L}(\mathcal{K}_X)/\mathcal{L}(\mathcal{O}_X)\big) \cong \Gamma\big(\mathcal{U}(\mathcal{K}_X)/\mathcal{U}(\mathcal{O}_X)\big)$. The rational map $u \in \mathcal{U}(\mathcal{K}_X)$, as a lift of ℓ_u, is determined up to a constant $c \in \mathcal{U}(k) = \Gamma\big(\mathcal{U}(\mathcal{O}_X)\big)$, according to the exact sequence

$$0 \longrightarrow \Gamma\big(\mathcal{U}(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(\mathcal{U}(\mathcal{K}_X)\big) \longrightarrow \Gamma\big(\mathcal{U}(\mathcal{K}_X)/\mathcal{U}(\mathcal{O}_X)\big).$$

For illustration, we can make this concrete as follows: If $\mathcal{D} \in \underline{\mathrm{Div}}^{0,\mathrm{red}}_X(S)$ is the divisor corresponding to $\lambda^\vee(\iota) \in \mathcal{F}(S) \subset \underline{\mathrm{Div}}^{0,\mathrm{red}}_X(S)$, then $\mathcal{O}_{X\otimes S}(\mathcal{D}) \in \mathrm{Pic}^{0,\mathrm{red}}_X(S)$ is the line bundle corresponding to $\mathcal{U} \in \mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathcal{L})$ under $\mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathcal{L}) \longrightarrow \mathrm{Ext}_{\mathcal{A}b/k}(\mathcal{A}, \mathbb{L}_S) = \mathrm{Pic}^0_{\mathcal{A}}(S) \xrightarrow{\sim} \mathrm{Pic}^{0,\mathrm{red}}_X(S)$. Then the rational map $u : X \dashrightarrow \mathcal{O}_{X\otimes S}(\mathcal{D})$ is the 1-section of $\mathcal{O}_{X\otimes S}(\mathcal{D})$, up to translation by a constant.

Step 2: Universality of $u : X \dashrightarrow \mathcal{U}$.
Let $\varphi : X \dashrightarrow G$ be a rational map to a smooth connected algebraic group G

with canonical decomposition $0 \to L \to G \xrightarrow{\rho} A \to 0$, inducing a homomorphism of formal groups $\tau_\varphi : L^\vee \longrightarrow \mathcal{F} \subset \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$, $\lambda \longmapsto \langle \lambda, \ell_\varphi \rangle$ (Definition 2.4). Let $l := (\tau_\varphi)^\vee : \mathcal{L} \longrightarrow L$ be the dual homomorphism of affine groups. The composition $X \overset{\varphi}{\dashrightarrow} G \xrightarrow{\rho} A$ extends to a morphism from X to an abelian variety. Translating φ by a constant $g \in G(k)$, if necessary, we may assume that $\rho \circ \varphi$ factors through $\mathcal{A} = \mathrm{Alb}(X)$:

$$\begin{tikzcd}
X \arrow[rr, "\rho\circ\varphi"] \arrow[dr, "\mathrm{alb}"'] & & A \\
& \mathrm{Alb}(X). \arrow[ur] &
\end{tikzcd}$$

We are going to show that we have a commutative diagram as follows:

$$\begin{tikzcd}
& \mathcal{L} \arrow[r, "l"] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[d] \\
& \mathcal{U} \arrow[r, "h"] \arrow[d] & l_*\mathcal{U} \arrow[r, "\sim"] \arrow[d] & G_{\mathcal{A}} \arrow[r] \arrow[d] & G \arrow[d, "\rho"] \\
X \arrow[ur, "u"] \arrow[r] \arrow[urrr, "\varphi_{\mathcal{A}}"] \arrow[urrrr, "\varphi"'] & \mathcal{A} \arrow[r, equal] & \mathcal{A} \arrow[r, equal] & \mathcal{A} \arrow[r] & A
\end{tikzcd}$$

where $G_{\mathcal{A}} = G \times_A \mathcal{A}$ is the fibre product and $l_*\mathcal{U} = \mathcal{U} \amalg_{\mathcal{L}} L$ the amalgamated sum. If $\bar{l} : \Gamma\big(\mathcal{L}(\mathcal{K}_X)/\mathcal{L}(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big)$ denotes the map induced by $l : \mathcal{L} \longrightarrow L$, then $\ell_{h\circ u} = \bar{l}(\ell_u)$. This yields

$$\begin{aligned}
\tau_{h\circ u} &= \langle ?, \ell_{h\circ u} \rangle \;=\; \big\langle ?, \bar{l}(\ell_u) \big\rangle \;=\; \langle ? \circ l, \ell_u \rangle \\
&= \tau_u \circ l^\vee \;=\; \tau_\varphi
\end{aligned}$$

since $\tau_u : \mathcal{F} \longrightarrow \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ is the inclusion by construction of u.
This implies that $l_*\mathcal{U}_X$ and G_X are isomorphic L-bundles over X. Then $l_*\mathcal{U}$ and $G_{\mathcal{A}}$ are isomorphic as extensions of $\mathcal{A}$ by L, using the isomorphism $\mathrm{Pic}^0_X \xrightarrow{\sim} \mathrm{Pic}^0_{\mathcal{A}}$. Thus $\tau_{h\circ u} = \tau_\varphi$ shows that $h \circ u$ and $\varphi_{\mathcal{A}}$ coincide up to translation. As $u : X \longrightarrow \mathcal{U}$ generates $\mathcal{U}$, each $h' : \mathcal{U} \longrightarrow G_{\mathcal{A}}$ fulfilling $h' \circ u = \varphi_{\mathcal{A}}$ coincides with h. Hence h is unique.

($\Longrightarrow$) Assume that $u : X \dashrightarrow \mathcal{U}$ is universal for **Mr**. Let $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ be the canonical decomposition of $\mathcal{U}$, and let $\mathcal{F}$ be the image of the induced transformation $\tau_u : \mathcal{L}^\vee \longrightarrow \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. For $\lambda \in \mathcal{L}^\vee(R)$ the uniqueness of the homomorphism $h_\lambda : \mathcal{U} \longrightarrow \lambda_*\mathcal{U}$ fulfilling $u_\lambda = h_\lambda \circ u$ implies that the rational maps $u_\lambda : X \dashrightarrow \lambda_*\mathcal{U}$ are non-isomorphic to each other for distinct

$\lambda \in \mathcal{L}^\vee(R)$. Hence $\mathrm{div}_R(u_{X,\nu}) \neq \mathrm{div}_R(u_{X,\lambda})$ for $\nu \neq \lambda \in \mathcal{L}^\vee(R)$. Therefore $\mathcal{L}^\vee \longrightarrow \mathcal{F}$ is injective, hence an isomorphism.

Let $\varphi : X \dashrightarrow G$ be an object of $\mathbf{Mr}$ and $0 \to L \to G \to A \to 0$ be the canonical decomposition of G. Translating φ by a constant $g \in G(k)$, if necessary, we may assume that $\varphi : X \dashrightarrow G$ factors through a unique homomorphism $h : \mathcal{U} \longrightarrow G$. The restriction of h to $\mathcal{L}$ gives a homomorphism of affine groups $l : \mathcal{L} \longrightarrow L$. Then the dual homomorphism $l^\vee : L^\vee \longrightarrow \mathcal{F}$ yields a factorization of $L^\vee \longrightarrow \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ through $\mathcal{F}$. Thus $\mathbf{Mr}$ is a subcategory of $\mathbf{Mr}_\mathcal{F}$. Now the properties $(\diamondsuit\, 1, 2)$ guarantee that $\mathbf{Mr}$ contains all rational maps which induce a transformation to $\mathcal{F}$, hence $\mathbf{Mr}$ is equivalent to $\mathbf{Mr}_\mathcal{F}$. ■

Notation 2.13. If $\mathcal{F}$ is a dual-algebraic formal group in $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$, then the universal object for $\mathbf{Mr}_\mathcal{F}$ is denoted by $\mathrm{alb}_\mathcal{F} : X \dashrightarrow \mathrm{Alb}_\mathcal{F}(X)$.

Remark 2.14. By construction, $\mathrm{Alb}_\mathcal{F}(X)$ is generated by X. Since X is reduced, $\mathrm{Alb}_\mathcal{F}(X)$ is reduced as well, thus smooth. In the proof of Theorem 2.12 we have seen that $\mathrm{Alb}_\mathcal{F}(X)$ is an extension of the abelian variety $\mathrm{Alb}(X)$ by the affine group $\mathcal{F}^\vee$. More precisely, $\left[0 \longrightarrow \mathrm{Alb}_\mathcal{F}(X)\right]$ is the dual 1-motive of $\left[\mathcal{F} \longrightarrow \mathrm{Pic}_X^{0,\mathrm{red}}\right]$. The rational map $\left(\mathrm{alb}_\mathcal{F} : X \dashrightarrow \mathrm{Alb}_\mathcal{F}(X)\right) \in \mathbf{Mr}_\mathcal{F}$ is characterized by the fact that the transformation $\tau_{\mathrm{alb}_\mathcal{F}} : L^\vee \longrightarrow \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ is the identity $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$.

2.3.2 Functoriality

Let $\mathcal{F} \subset \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ be a dual-algebraic formal group. Let $\psi : Y \longrightarrow X$ be a morphism of smooth proper varieties, such that no irreducible component of $\psi(Y)$ is contained in $\mathrm{Supp}(\mathcal{F})$. For each dual-algebraic formal group $\mathcal{G} \subset \underline{\mathrm{Div}}_Y^{0,\mathrm{red}}$ containing $\psi^*\mathcal{F}$ the pull-back of relative Cartier divisors and of line bundles induces a homomorphism of 1-motives

$$\left[\mathcal{G} \to \mathrm{Pic}_Y^{0,\mathrm{red}}\right] \longleftarrow \left[\mathcal{F} \to \mathrm{Pic}_X^{0,\mathrm{red}}\right].$$

According to the construction of universal objects (Remark 2.14), we obtain via dualization of 1-motives

Proposition 2.15. *Let $\mathcal{F} \subset \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ be a dual-algebraic formal group. Let $\psi : Y \longrightarrow X$ be a morphism of smooth proper varieties, such that no irreducible component of $\psi(Y)$ is contained in* $\mathrm{Supp}(\mathcal{F})$. *Then ψ induces a homomorphism of algebraic groups*

$$\mathrm{Alb}_\mathcal{G}^\mathcal{F}(\psi) : \mathrm{Alb}_\mathcal{G}(Y) \longrightarrow \mathrm{Alb}_\mathcal{F}(X)$$

for each formal group $\mathcal{G} \subset \underline{\mathrm{Div}}_Y^{0,\mathrm{red}}$ *containing* $\psi^*\mathcal{F}$.

2.3.3 Descent of the Base Field

Let k be a perfect field. Let $\overline{k}$ be an algebraic closure of k. Let X be a smooth proper variety defined over k, write $\overline{X} = X \otimes_k \overline{k}$. Let $\mathcal{F} \subset \underline{\mathrm{Div}}_{\overline{X}}^0$ be a formal group, and suppose that $\mathcal{F}$ is defined over k:

Definition 2.16. A formal group $\mathcal{F} \subset \underline{\mathrm{Div}}_{\overline{X}}^0$ is *defined over* k, if for each finitely generated $\overline{k}$-algebra R the following condition is satisfied:
If $\mathcal{D} \in \underline{\mathrm{Div}}_{\overline{X}}^0(R)$ then for each $\sigma \in \mathrm{Gal}(\overline{k}/k)$ it holds $\mathcal{D}^\sigma \in \underline{\mathrm{Div}}_{\overline{X}}^0(R)$, where $\mathcal{D}^\sigma$ is the conjugate of $\mathcal{D}$ by means of σ.

The wish is to show that the universal object $\mathrm{alb}_{\mathcal{F}} : \overline{X} \dashrightarrow \mathrm{Alb}_{\mathcal{F}}(\overline{X})$ for the category $\mathbf{Mr}_{\mathcal{F}}$ can be defined over k. The main tool for this purpose will be the method of *Galois descent*, as described in [Ser3, V, § 4]. A short sketch of this procedure is the following: Let k_1 be a (finite) Galois extension of k. Let V_1 be a k_1-variety living in a category $\mathfrak{C}$ of k_1-varieties (e.g. algebraic k_1-groups or k_1-torsors). Suppose we are given $\mathfrak{C}$-isomorphisms $h_\sigma : V_1 \longrightarrow V_1^\sigma$ between V_1 and its conjugate V_1^σ for each $\sigma \in \mathrm{Gal}(k_1/k)$. If V_1 is a homogeneous space for an algebraic k_1-group or satisfies certain other criteria (see [Ser3, V, No. 20]) and the h_σ satisfy the identity

$$h_{\sigma\tau} = (h_\sigma)^\tau \circ h_\tau$$

then there exists a k-variety V and a $\mathfrak{C}$-isomorphism $f : V \otimes_k k_1 \xrightarrow{\sim} V_1$. Here V inherits the structure of V_1 preserved under the h_σ. In this case it holds $h_\sigma = f^\sigma \circ f^{-1}$.

When one does not assume that X is endowed with a k-rational point, one is led to two different descents of $\mathrm{Alb}_{\mathcal{F}}(\overline{X})$:

(1) The universal mapping property of $\mathrm{alb}_{\mathcal{F}} : \overline{X} \dashrightarrow \mathrm{Alb}_{\mathcal{F}}(\overline{X})$ gives transformations $h_\sigma^{(1)}$ which are "affine homomorphisms", i.e. compositions of a homomorphism by a translation. Therefore the descent of $\mathrm{Alb}_{\mathcal{F}}(\overline{X})$ by means of the $h_\sigma^{(1)}$ yields a k-torsor $\mathrm{Alb}_{\mathcal{F}}^{(1)}(X)$.

(2) In order to avoid translations or the reference to base points, one may reformulate the universal mapping property, replacing rational maps $\varphi : X \dashrightarrow G$ from $\overline{X}$ to algebraic groups by its associated "difference maps" $\varphi^{(-)} : \overline{X} \times \overline{X} \dashrightarrow G$, $(p,q) \longmapsto \varphi(q) - \varphi(p)$. In this way translations are eliminated and one obtains transformations $h_\sigma^{(0)}$ which are homomorphisms of algebraic groups. Then the descent of $\mathrm{Alb}_{\mathcal{F}}(\overline{X})$ by means of the $h_\sigma^{(0)}$ yields

an algebraic k-group $\mathrm{Alb}_{\mathcal{F}}^{(0)}(X)$. This is the k-group acting on the k-torsor $\mathrm{Alb}_{\mathcal{F}}^{(1)}(X)$.

Notation 2.17. If $\varphi : \overline{X} \dashrightarrow P$ is a rational map to a torsor (= principal homogeneous space) P for a group G, then

$$\varphi^{(-)} : \overline{X} \times \overline{X} \dashrightarrow G$$

denotes the rational map to the group G which assigns to $(p,q) \in \overline{X} \times \overline{X}$ the unique $g \in G$ such that $g \cdot \varphi(p) = \varphi(q)$.

Notation 2.18. If $\varphi : \overline{X} \dashrightarrow P$ is a rational map to a torsor, then set

$$\varphi^{(1)} := \varphi\,, \qquad \varphi^{(0)} := \varphi^{(-)}.$$

Remark 2.19. If a k-torsor P for an algebraic k-group G admits a k-rational point, then P may be identified with G. Then for a rational map $\varphi : X \dashrightarrow P$ it makes sense to consider the base changed map $\varphi \otimes_k \overline{k} : X \otimes_k \overline{k} \dashrightarrow P \otimes_k \overline{k}$ as a rational map from $X \otimes_k \overline{k} = \overline{X}$ to an algebraic $\overline{k}$-group $P \otimes_k \overline{k} \cong G \otimes_k \overline{k}$.

Theorem 2.20. *There exists a k-torsor $\mathrm{Alb}_{\mathcal{F}}^{(1)}(X)$ for an algebraic k-group $\mathrm{Alb}_{\mathcal{F}}^{(0)}(X)$ and rational maps defined over k*

$$\mathrm{alb}_{\mathcal{F}}^{(i)} : X^{2-i} \dashrightarrow \mathrm{Alb}_{\mathcal{F}}^{(i)}(X)$$

for $i = 1, 0$, satisfying the following universal property:

If $\varphi : X \dashrightarrow G^{(1)}$ is a rational map defined over k to a k-torsor $G^{(1)}$ for an algebraic k-group $G^{(0)}$, such that $\varphi \otimes_k \overline{k}$ is an object of $\mathbf{Mr}_{\mathcal{F}}\left(\overline{X}\right)$, then there exist a unique affine homomorphism of k-torsors $h^{(1)}$ and a unique homomorphism of algebraic k-groups $h^{(0)}$, $h^{(i)} : \mathrm{Alb}_{\mathcal{F}}^{(i)}(X) \longrightarrow G^{(i)}$, defined over k, such that $\varphi^{(i)} = h^{(i)} \circ \mathrm{alb}_{\mathcal{F}}^{(i)}$ for $i = 1, 0$.

Proof. The same arguments as given in [Ser3, V, No. 22] work in our situation. ■

3 Albanese with Modulus

Let X be a smooth proper variety over k, an algebraically closed field of arbitrary characteristic. Let D be an effective divisor on X (with multiplicity). The Albanese $\mathrm{Alb}(X,D)$ of X of modulus D is a higher dimensional analogon to the generalized Jacobian with modulus of Rosenlicht-Serre. $\mathrm{Alb}(X,D)$ is

defined by the universal mapping property for morphisms from $X \setminus D$ to algebraic groups of modulus $\leq D$ (Definition 3.11). Our definition of the modulus of rational maps into algebraic groups coincides with the classical definition from [Ser3, III, § 1] in the curve case. Therefore the Albanese with modulus agrees with the Jacobian with modulus of Rosenlicht-Serre for curves, which we review in Subsection 3.3.

In Subsection 3.4 we consider a Chow group $\mathrm{CH}_0(X,D)^0$ of 0-cycles relative to the modulus D (Definition 3.28). We give an alternative characterization of $\mathrm{Alb}(X,D)$ as a universal quotient of $\mathrm{CH}_0(X,D)^0$ (Theorem 3.30).

In Subsection 3.5 we give an application to class field theory of function fields of varieties over finite fields. We can rephrase Lang's class field theory by replacing *maximal maps* by *Albanese varieties with modulus*.

3.1 Witt-related Topics

Here we recall and state some purely technical notions that are needed for the construction of the Albanese with modulus. No. 3.1.1 is folklore, while No. 3.1.2 is a global version of some basic notions from [KR2]. It is possible for the reader to skip this subsection and trace back the definitions when needed.

3.1.1 Artin-Hasse Exponential

Suppose $\mathrm{char}(k) = p > 0$. Let E be the series

$$\mathrm{E}\,(t) \;=\; \exp\left(-\sum_{r\geq 0}\frac{t^{p^r}}{p^r}\right) \;=\; \prod_{\substack{r\geq 1\\ (r,p)=1}} \left(1-t^r\right)^{\mu(r)/r}$$

where μ denotes the Möbius function, i.e.

$$\mu\,(r) = \begin{cases} 0 & \text{if } r \text{ is divisible by the square of a prime,}\\ (-1)^n & \text{if } r = p_1\cdots p_n \text{ and } p_1,\ldots,p_n \text{ are distinct primes,}\\ 1 & \text{if } r = 1.\end{cases}$$

Let $w = (w_0, w_1, \ldots, w_r, \ldots)$ be a Witt vector (of finite or infinite length). The product series

$$\mathrm{Exp}\,(w) \;=\; \prod_{r\geq 0}\mathrm{E}\,(w_r)$$

satisfies for Witt vectors w, v

$$\operatorname{Exp}(w+v) \;=\; \operatorname{Exp}(w)\cdot \operatorname{Exp}(v)\,.$$

Exp is called *Artin-Hasse exponential.*

For details see for example [Dem, III, No.s 1 and 2].

3.1.2 Filtrations of the Witt Group

Let (K, v) be a discrete valuation field of characteristic $p > 0$ with residue field k. We define filtrations $\mathrm{fil}_n \mathbb{W}_r(K)$ and $\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(K)$, $n \in \mathbb{N}$, on the group $\mathbb{W}_r(K)$ of Witt vectors of length r.

Definition 3.1. Let $\mathrm{fil}_n \mathbb{W}_r(K)$ be the following subgroup of $\mathbb{W}_r(K)$:

$$\mathrm{fil}_n \mathbb{W}_r(K) = \left\{ (f_{r-1}, \ldots, f_0) \;\middle|\; \begin{array}{l} f_i \in K, \quad \mathrm{v}\,(f_i) \geq -n/p^i \\ \forall\, 0 \leq i \leq r-1 \end{array} \right\},$$

cf. [Bry, No. 1, Prop. 1]. Let $\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(K)$ be the subgroup of $\mathbb{W}_r(K)$ generated by $\mathrm{fil}_n \mathbb{W}_r(K)$ by means of the Frobenius F, cf. [KR2, 2.2],

$$\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(K) = \sum_{\nu \geq 0} \mathrm{F}^{\nu}\, \mathrm{fil}_n \mathbb{W}_r(K).$$

Let X be a variety over k, regular in codimension 1. Let $D = \sum_{q \in S} n_q D_q$ be an effective divisor on X, where S is a finite set of points of codimension 1 in X, where D_q are the prime divisors associated to $q \in S$ and n_q are integers ≥ 1 for $q \in S$.

Definition 3.2. Let $\mathrm{fil}_D \mathbb{W}_r(\mathcal{K}_X)$ (resp. $\mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)$) be the sheaf of subgroups of $\mathbb{W}_r(\mathcal{K}_X)$ formed by the groups

$$\big(\mathrm{fil}_D \mathbb{W}_r(\mathcal{K}_X)\big)(U) = \left\{ w \in \mathbb{W}_r(\mathcal{K}_X) \;\middle|\; \begin{array}{ll} w \in \mathrm{fil}_{n_q} \mathbb{W}_r(\mathcal{K}_{X,q}) & \forall q \in S \cap U \\ w \in \mathbb{W}_r(\mathcal{O}_{X,p}) & \forall p \in U \setminus S \end{array} \right\}$$

resp.

$$\big(\mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)\big)(U) = \left\{ w \in \mathbb{W}_r(\mathcal{K}_X) \;\middle|\; \begin{array}{ll} w \in \mathrm{fil}^{\mathrm{F}}_{n_q} \mathbb{W}_r(\mathcal{K}_{X,q}) & \forall q \in S \cap U \\ w \in \mathbb{W}_r(\mathcal{O}_{X,p}) & \forall p \in U \setminus S \end{array} \right\}$$

for open $U \in X$, where $\mathrm{fil}_n \mathbb{W}_r(\mathcal{K}_{X,q})$ (resp. $\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(\mathcal{K}_{X,q})$) denotes the filtration associated to the valuation v_q attached to the point $q \in S$.

Proposition 3.3. *Suppose X is a projective variety over k and D an effective divisor on X. Then $\Gamma\big(X, \mathrm{fil}_D \mathbb{W}_r(\mathcal{K}_X)\big)$ is a finite $\mathbb{W}_r(k)$-module.*

Proof. The Verschiebung $\mathrm{V} : \mathbb{W}_{r-1} \longrightarrow \mathbb{W}_r$ yields an exact sequence

$$0 \longrightarrow \mathrm{fil}_D \mathbb{W}_{r-1}(\mathcal{K}_X) \longrightarrow \mathrm{fil}_D \mathbb{W}_r(\mathcal{K}_X) \longrightarrow \mathrm{fil}_{\lfloor D/p^{r-1} \rfloor} \mathbb{W}_1(\mathcal{K}_X) \longrightarrow 0$$

where $\lfloor D/p^{r-1} \rfloor = \sum_{q \in S} \lfloor n_q/p^{r-1} \rfloor D_q$. This induces the exact sequence

$$0 \longrightarrow \Gamma\big(\mathrm{fil}_D \mathbb{W}_{r-1}(\mathcal{K}_X)\big) \longrightarrow \Gamma\big(\mathrm{fil}_D \mathbb{W}_r(\mathcal{K}_X)\big) \longrightarrow \Gamma\big(\mathrm{fil}_{\lfloor D/p^{r-1} \rfloor} \mathbb{W}_1(\mathcal{K}_X)\big).$$

By induction over $r \geq 1$ and since $\mathbb{W}_1(k) = k$ is noetherian, it is sufficient to show the statement for $r = 1$. Now $\mathrm{fil}_D \mathbb{W}_1(\mathcal{K}_X) = \mathcal{O}_X(D)$ is a coherent sheaf, hence $\Gamma\big(X, \mathrm{fil}_D \mathbb{W}_1(\mathcal{K}_X)\big)$ is a finite module over $\mathbb{W}_1(k) = k$. ■

Definition 3.4. Let R be a commutative ring over $\mathbb{F}_p$. We let $R[\mathrm{F}]$ be the non-commutative polynomial ring defined by

$$R[\mathrm{F}] = \left\{ \sum_{i=1}^{n} \mathrm{F}^i\, r_i \;\middle|\; \begin{matrix} r_i \in R \\ n \in \mathbb{N} \end{matrix} \right\}, \qquad\qquad \mathrm{F}\, r = r^p\, \mathrm{F} \qquad \forall r \in R.$$

Definition 3.5. If $\Omega_{\mathcal{K}_X} = \Omega_{\mathcal{K}_X/k}$ is the module of differentials of $\mathcal{K}_X$ over k, we let δ be the homomorphism

$$\delta : \mathbb{W}_r(\mathcal{K}_X) \longrightarrow \Omega_{\mathcal{K}_X}, \qquad\qquad (f_{r-1}, \dots, f_0) \longmapsto \sum_{i=0}^{r-1} f_i^{p^i - 1} \mathrm{d} f_i.$$

Definition 3.6. If E is a reduced effective divisor on X with normal crossings, we let $\Omega_X(\log E)$ be the sheaf of differentials on X with log-poles along E, i.e. the $\mathcal{O}_X$-module generated locally by $\mathrm{d}f$, $f \in \mathcal{O}_X$ and $\mathrm{d}\log t = t^{-1}\mathrm{d}t$, where t is a local equation for E.

Proposition 3.7. *Suppose D_{red} is a normal crossing divisor. The homomorphism δ from Definition 3.5 induces injective homomorphisms*

$$\begin{aligned} \mathfrak{d}_D :\ \mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)/\mathrm{fil}^{\mathrm{F}}_{\lfloor D/p \rfloor} \mathbb{W}_r(\mathcal{K}_X) &\longrightarrow \mathfrak{D}_D \\ \overline{\mathfrak{d}}_D :\ \mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)/\mathrm{fil}^{\mathrm{F}}_{D - D_{\mathrm{red}}} \mathbb{W}_r(\mathcal{K}_X) &\longrightarrow \overline{\mathfrak{D}}_D \end{aligned}$$

where $\mathfrak{D}_D$ resp. $\overline{\mathfrak{D}}_D$ are the $\mathcal{O}_X$-modules

$$\begin{aligned} \mathfrak{D}_D &= k[\mathrm{F}] \otimes_k \big(\Omega_X(\log D_{\mathrm{red}}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)/\mathcal{O}_X(\lfloor D/p \rfloor)\big), \\ \overline{\mathfrak{D}}_D &= k[\mathrm{F}] \otimes_k \big(\Omega_X(\log D_{\mathrm{red}}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)/\mathcal{O}_X(D - D_{\mathrm{red}})\big). \end{aligned}$$

and $\lfloor D/p \rfloor$ means the largest divisor E such that $pE \leq D$.

Proof. This is the global formulation of [KR2, 4.6]. ■

Definition 3.8. Let ${}^\flat\overline{\mathfrak{D}}_D$ be the image in $\overline{\mathfrak{D}}_D$ of the $\mathcal{O}_X$-module $k[\mathrm{F}] \otimes_k \big(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)/\mathcal{O}_X(D - D_{\mathrm{red}})\big)$ (without log-poles). Then

$$ {}^\flat\overline{\mathfrak{D}}_D \cong k[\mathrm{F}] \otimes_k \big(\Omega_{D_{\mathrm{red}}} \otimes_{\mathcal{O}_{D_{\mathrm{red}}}} \mathcal{O}_X(D)/\mathcal{O}_X(D - D_{\mathrm{red}})\big) $$

since $t^{-n_q}\mathrm{d}t = t^{1-n_q}\mathrm{d}\log t$ vanishes in $\overline{\mathfrak{D}}_D$ for any local equation t of D_{red}. Then we let ${}^\flat\mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X) \subset \mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)$ be the inverse image of ${}^\flat\overline{\mathfrak{D}}_D$ under the map $\overline{\mathfrak{d}}_D$ from Proposition 3.7. According to [KR2, 4.7], this is a global version of the following alternative

Definition 3.9. Let ${}^\flat\mathrm{fil}_n \mathbb{W}_r(K)$ be the subgroup of $\mathrm{fil}_n \mathbb{W}_r(K)$ consisting of all elements $(f_{r-1}, \ldots, f_0)$ satisfying the following condition: If the p-adic order ν of n is $< r$, then $p^\nu\, \mathrm{v}(f_\nu) > -n$. Then ${}^\flat\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(K)$ is the subgroup of $\mathbb{W}_r(K)$ generated by ${}^\flat\mathrm{fil}_n \mathbb{W}_r(K)$ by means of the Frobenius F,

$$ {}^\flat\mathrm{fil}^{\mathrm{F}}_n \mathbb{W}_r(K) = \sum_{\nu \geq 0} \mathrm{F}^\nu\ {}^\flat\mathrm{fil}_n \mathbb{W}_r(K). $$

Lemma 3.10. *Let $\psi : Y \longrightarrow X$ be a morphism of varieties over k, such that $\psi(Y)$ intersects* $\mathrm{Supp}(D)$ *properly. Let $D \cdot Y$ denote the pull-back of D to Y. Suppose that D_{red} and $(D \cdot Y)_{\mathrm{red}}$ are normal crossing divisors. There is a commutative diagram of homomorphisms with injective rows*

$$ \begin{array}{ccc} \mathrm{fil}^{\mathrm{F}}_D \mathbb{W}_r(\mathcal{K}_X)/\mathrm{fil}^{\mathrm{F}}_{\lfloor D/p \rfloor} \mathbb{W}_r(\mathcal{K}_X) & \xrightarrow{\mathfrak{d}_{X,D}} & \mathfrak{D}_{X,D} \\ \big\downarrow & & \big\downarrow \\ \mathrm{fil}^{\mathrm{F}}_{D\cdot Y} \mathbb{W}_r(\mathcal{K}_Y)/\mathrm{fil}^{\mathrm{F}}_{\lfloor D\cdot Y/p \rfloor} \mathbb{W}_r(\mathcal{K}_Y) & \xrightarrow{\mathfrak{d}_{Y,D\cdot Y}} & \mathfrak{D}_{Y,D\cdot Y} \end{array} $$

where the vertical arrows are the obvious pull-back maps from X to Y.

Proof. Straightforward. ■

3.2 Albanese with Modulus

3.2.1 Existence and Construction

Let X be a smooth proper variety over k, which is an algebraically closed field of arbitrary characteristic, unless stated otherwise.

Definition 3.11. Let $\varphi : X \dashrightarrow G$ be a rational map from X to a smooth connected algebraic group G. Let L be the affine part of G and U the unipotent part of L. We define an effective divisor, called the *modulus of* φ

$$\operatorname{mod}(\varphi) = \sum_{\operatorname{ht}(q)=1} \operatorname{mod}_q(\varphi)\ D_q$$

where q ranges over all points of codimension 1 in X, and D_q is the prime divisor associated to q. For each $q \in X$ of codimension 1, the canonical map $L(\mathcal{K}_{X,q})/L(\mathcal{O}_{X,q}) \longrightarrow G(\mathcal{K}_{X,q})/G(\mathcal{O}_{X,q})$ is bijective, cf. 2.2.1. Take an element $l_q \in L(\mathcal{K}_{X,q})$ whose image in $G(\mathcal{K}_{X,q})/G(\mathcal{O}_{X,q})$ coincides with the class of $\varphi \in G(\mathcal{K}_{X,q})$. If $\operatorname{char}(k) = 0$, let $(u_{q,i})_{1\le i\le s}$ be the image of l_q in $\mathbb{G}_\mathrm{a}(\mathcal{K}_{X,q})^s$ under $L \to U \cong (\mathbb{G}_\mathrm{a})^s$. If $\operatorname{char}(k) = p > 0$, let $(u_{q,i})_{1\le i\le s}$ be the image of l_q in $\mathbb{W}_r(\mathcal{K}_{X,q})^s$ under $L \to U \subset (\mathbb{W}_r)^s$.

$$\operatorname{mod}_q(\varphi) = \begin{cases} 0 & \text{if } \ \varphi \in G(\mathcal{O}_{X,q}) \\ 1 + \max\{\mathrm{n}_q(u_{q,i}) \mid 1 \le i \le s\} & \text{if } \ \varphi \notin G(\mathcal{O}_{X,q}) \end{cases}$$

where for $u \in \mathbb{G}_\mathrm{a}(\mathcal{K}_{X,q})$ resp. $\mathbb{W}_r(\mathcal{K}_{X,q})$

$$\mathrm{n}_q(u) = \begin{cases} -\mathrm{v}_q(u) & \text{if } \operatorname{char}(k) = 0 \\ \min\left\{n \in \mathbb{N} \mid u \in \operatorname{fil}^{\mathrm{F}}_n \mathbb{W}_r(\mathcal{K}_{X,q})\right\} & \text{if } \operatorname{char}(k) = p > 0. \end{cases}$$

Note that $\operatorname{mod}_v(\varphi)$ is independent of the choice of the isomorphism $U \cong (\mathbb{G}_\mathrm{a})^s$ resp. of the embedding $U \subset (\mathbb{W}_r)^s$, see [KR2, Thm. 3.3].

Definition 3.12. Let D be an effective divisor on X. Then $\mathbf{Mr}^{X,D}$ denotes the category of those rational maps φ from X to algebraic groups such that $\operatorname{mod}(\varphi) \le D$. The universal object of $\mathbf{Mr}^{X,D}$ (if it exists) is denoted by $\operatorname{Alb}(X,D)$ and called the *Albanese of* X *of modulus* D.

Definition 3.13. Let D be an effective divisor on X, let D_red be the reduced part of D. Then $\mathcal{F}_{X,D}$ denotes the formal subgroup of $\underline{\mathrm{Div}}_X$ characterized by

$$(\mathcal{F}_{X,D})_{\text{ét}} = \left\{B \in \underline{\mathrm{Div}}_X(k) \;\middle|\; \operatorname{Supp}(B) \subset \operatorname{Supp}(D)\right\}$$

and for $\operatorname{char}(k) = 0$

$$(\mathcal{F}_{X,D})_\mathrm{inf} = \exp\Big(\widehat{\mathbb{G}}_\mathrm{a} \otimes_k \Gamma\big(\mathcal{O}_X(D - D_\mathrm{red})/\mathcal{O}_X\big)\Big)$$

for $\operatorname{char}(k) = p > 0$

$$(\mathcal{F}_{X,D})_\mathrm{inf} = \operatorname{Exp}\left(\sum_{r>0} {}_r\widehat{\mathbb{W}} \otimes_{\mathbb{W}_r(k)} \Gamma\Big(\operatorname{fil}^{\mathrm{F}}_{D-D_\mathrm{red}} \mathbb{W}_r(\mathcal{K}_X)\Big/\mathbb{W}_r(\mathcal{O}_X)\Big)\right).$$

Let $\mathcal{F}^{0,\mathrm{red}}_{X,D} = \mathcal{F}_{X,D} \times_{\underline{\mathrm{Div}}_X} \underline{\mathrm{Div}}^{0,\mathrm{red}}_X$ be the intersection of $\mathcal{F}_{X,D}$ and $\underline{\mathrm{Div}}^{0,\mathrm{red}}_X$.

Proposition 3.14. *The formal groups $\mathcal{F}_{X,D}$ and $\mathcal{F}^{0,\mathrm{red}}_{X,D}$ are dual-algebraic.*

Proof. The statement is obvious for $\mathrm{char}(k) = 0$, therefore we suppose $\mathrm{char}(k) = p > 0$. The proof is done in two steps. Let $\mathcal{F}^{\lrcorner}_{X,D}$ be the formal subgroup of $\underline{\mathrm{Div}}_X$ defined in the same way as $\mathcal{F}_{X,D}$, but using the filtration $\mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)$ instead of $\mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)$. In the first step, we show that for any effective divisor D the formal group $\mathcal{F}^{\lrcorner}_{X,D}$ is dual-algebraic. In the second step, we show that for any D there exists $D' \geq D$ such that $\mathcal{F}_{X,D}$ is contained in the image of $\mathcal{F}^{\lrcorner}_{X,D'}$ in $\mathcal{F}_{X,D'}$. Thus $\mathcal{F}_{X,D}$ is a formal subgroup of a quotient of a dual-algebraic formal group, hence dual-algebraic by Lemma 3.15. Then also the formal subgroup $\mathcal{F}^{0,\mathrm{red}}_{X,D}$ of $\mathcal{F}_{X,D}$ is dual-algebraic.

Step 1: Let D be an effective divisor on X. Write $D = \sum_{\mathrm{ht}(q)=1} n_q\, D_q$, where q ranges over all points of codimension 1 in X, and D_q is the prime divisor associated to q. Let S be the finite set of those q with $n_q > 0$. Let $m = \min\{r \mid p^r > n_q - 1\ \forall q \in S\}$. Hence for $r \geq m$, if $(f_{r-1}, \ldots, f_0) \in \mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)$, then $f_i \in \mathcal{O}_X$ for $r > i \geq m$, according to Definition 3.2. Then the Verschiebung $\mathrm{V}^{r-m} : \mathbb{W}_m(\mathcal{K}_X) \longrightarrow \mathbb{W}_r(\mathcal{K}_X)$ yields a surjective homomorphism $\mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_m(\mathcal{K}_X)/\mathbb{W}_m(\mathcal{O}_X) \twoheadrightarrow \mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)$. Thus $\left(\mathcal{F}^{\lrcorner}_{X,D}\right)_{\mathrm{inf}}$ is already generated by a finite sum via Exp:

$$\left(\mathcal{F}^{\lrcorner}_{X,D}\right)_{\mathrm{inf}} = \mathrm{Exp}\left(\sum_{1\leq r\leq m} {}_r\widehat{\mathbb{W}} \otimes_{\mathbb{W}_r(k)} \Gamma\Big(\mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)\Big/\mathbb{W}_r(\mathcal{O}_X)\Big)\right).$$

Each $\Gamma\big(X, \mathrm{fil}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)$ is a finitely generated $\mathbb{W}_r(k)$-module, by the same proof as for Proposition 3.3. Hence $\left(\mathcal{F}^{\lrcorner}_{X,D}\right)_{\mathrm{inf}}$ is a quotient of the direct sum of finitely many ${}_r\widehat{\mathbb{W}}$.

Moreover, $\left(\mathcal{F}^{\lrcorner}_{X,D}\right)_{\mathrm{ét}} = (\mathcal{F}_{X,D})_{\mathrm{ét}}$ is an abelian group of finite type, since D has only finitely many components. Thus $\mathcal{F}^{\lrcorner}_{X,D}$ is dual-algebraic, according to Proposition 1.21.

Step 2: We show that for any effective divisor D there exists an effective divisor $D' \geq D$ such that $\mathcal{F}_{X,D}$ is generated by $\mathcal{F}^{\lrcorner}_{X,D'}$. We will find an effective divisor $D' \geq D$ such that $\Gamma\big(\mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)$ is generated by $\sum_{\nu\geq 0} \mathrm{F}^{\nu}\,\Gamma\big(\mathrm{fil}_{D'-D'_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)$. Since the homomorphism $\mathrm{V}^{r-m} : \mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}}\mathbb{W}_m(\mathcal{K}_X)/\mathbb{W}_m(\mathcal{O}_X) \longrightarrow \mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)$ is surjective for $r \geq m$, we only need to consider $r = m$. This is sufficient because $\mathrm{Exp}\,(v \otimes \sum_i \mathrm{F}^{\nu_i}\,\omega_i) = \mathrm{Exp}\,(\sum_i \mathrm{V}^{\nu_i}\, v \otimes \omega_i)$.

The exact sequence

$$0 \longrightarrow \mathbb{W}_r(\mathcal{O}_X) \longrightarrow \mathbb{W}_r(\mathcal{K}_X) \longrightarrow \mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X) \longrightarrow 0$$

yields the exact sequence

$$\Gamma\big(\mathbb{W}_r(\mathcal{K}_X)\big) \longrightarrow \Gamma\big(\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big) \longrightarrow \mathrm{H}^1\big(\mathbb{W}_r(\mathcal{O}_X)\big) \longrightarrow 0.$$

Here $\mathrm{H}^1\big(\mathbb{W}_r(\mathcal{K}_X)\big) = 0$ since $\mathbb{W}_r(\mathcal{K}_X)$ is a flasque sheaf. Since $\mathrm{H}^1\big(\mathbb{W}_r(\mathcal{O}_X)\big)$ is a finite $\mathbb{W}_r(k)$-module, there is an effective divisor E such that the map $\Gamma\big(\mathrm{fil}_E\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big) \longrightarrow \mathrm{H}^1\big(\mathbb{W}_r(\mathcal{O}_X)\big)$ is surjective. Hence for any $\sigma \in \Gamma\big(\mathrm{fil}^{\mathrm{F}}_{D-D_{\mathrm{red}}}\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)$ there is $\rho \in \Gamma\big(\mathrm{fil}_E\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)$ such that $\sigma - \rho$ lies in the image of $\Gamma\big(\mathbb{W}_r(\mathcal{K}_X)\big)$, hence in the image of $\Gamma\big(\mathrm{fil}_{E'}\mathbb{W}_r(\mathcal{K}_X)\big)$, where $E' = \max\{E, D - D_{\mathrm{red}}\}$. Therefore we are reduced to showing that for any D there exists $D' \geq D$ such that $\Gamma\big(\mathrm{fil}^{\mathrm{F}}_D\mathbb{W}_r(\mathcal{K}_X)\big)$ is generated by $\sum_{\nu\geq 0} \mathrm{F}^\nu\,\Gamma\big(\mathrm{fil}_{D'}\mathbb{W}_r(\mathcal{K}_X)\big)$.

Consider the exact sequence

$$0 \longrightarrow \bigoplus_{\nu\geq 0} \mathrm{fil}_{\lfloor D/p\rfloor}\mathbb{W}_r(\mathcal{K}_X) \longrightarrow \bigoplus_{\nu\geq 0} \mathrm{fil}_D\mathbb{W}_r(\mathcal{K}_X) \longrightarrow \mathrm{fil}^{\mathrm{F}}_D\mathbb{W}_r(\mathcal{K}_X) \longrightarrow 0$$

where the third arrow is $(w_\nu)_\nu \longmapsto \sum_\nu \mathrm{F}^\nu w_\nu$, and the second arrow is $(w_\nu)_\nu \longmapsto (\mathrm{F}\,w_\nu - w_{\nu-1})_\nu$, where we set $w_{-1} = 0$. Here $\lfloor D/p\rfloor$ means the largest divisor E such that $pE \leq D$. This yields an exact sequence

$$\bigoplus_{\nu\geq 0}\Gamma\big(\mathrm{fil}_D\mathbb{W}_r(\mathcal{K}_X)\big) \longrightarrow \Gamma\big(\mathrm{fil}^{\mathrm{F}}_D\mathbb{W}_r(\mathcal{K}_X)\big) \longrightarrow \bigoplus_{\nu\geq 0}\mathrm{H}^1\big(\mathrm{fil}_{\lfloor D/p\rfloor}\mathbb{W}_r(\mathcal{K}_X)\big).$$

$\mathbb{W}_r(\mathcal{K}_X)$ is the inductive limit of $\mathrm{fil}_E\mathbb{W}_r(\mathcal{K}_X)$, where E ranges over all effective divisors on X, hence

$$0 = \mathrm{H}^1\big(\mathbb{W}_r(\mathcal{K}_X)\big) = \mathrm{H}^1\left(\varinjlim_E \mathrm{fil}_E\mathbb{W}_r(\mathcal{K}_X)\right) = \varinjlim_E \mathrm{H}^1\big(\mathrm{fil}_E\mathbb{W}_r(\mathcal{K}_X)\big).$$

As $\mathrm{H}^1\big(\mathrm{fil}_{\lfloor D/p\rfloor}\mathbb{W}_r(\mathcal{K}_X)\big)$ is a finite $\mathbb{W}_r(k)$-module, there is an effective divisor $D' \geq D$ such that the image of $\mathrm{H}^1\big(\mathrm{fil}_{\lfloor D/p\rfloor}\mathbb{W}_r(\mathcal{K}_X)\big)$ in $\mathrm{H}^1\big(\mathrm{fil}_{\lfloor D'/p\rfloor}\mathbb{W}_r(\mathcal{K}_X)\big)$ is 0. Thus the image of $\Gamma\big(\mathrm{fil}^{\mathrm{F}}_D\mathbb{W}_r(\mathcal{K}_X)\big)$ in $\Gamma\big(\mathrm{fil}^{\mathrm{F}}_{D'}\mathbb{W}_r(\mathcal{K}_X)\big)$ is contained in $\sum_{\nu\geq 0}\mathrm{F}^\nu\,\Gamma\big(\mathrm{fil}_{D'}\mathbb{W}_r(\mathcal{K}_X)\big)$. ■

Lemma 3.15. *Let $\mathcal{F}$ be a dual-algebraic formal group. Then any formal group $\mathcal{G}$ that is a subgroup or a quotient of $\mathcal{F}$ is also dual-algebraic.*

Proof. By Cartier-duality, this is equivalent to Lemma 3.16 below. ■

Lemma 3.16. *Let L be an affine algebraic group. Then any affine closed subgroup K of L and any affine quotient N of L is also algebraic.*

Proof. [Dem, II, No. 6, Cor. 4 of Thm. 2]. ■

Lemma 3.17. *Let $\varphi : X \dashrightarrow G$ be a rational map from X to a smooth connected algebraic group G. Then the following conditions are equivalent:*

(i) $\operatorname{mod}(\varphi) \le D$,

(ii) $\operatorname{im}(\tau_\varphi) \subset \mathcal{F}_{X,D}$.

Proof. Write $D = \sum_{\mathrm{ht}(q)=1} n_q\, D_q$, where q ranges over all points in X of codimension 1, and D_q is the prime divisor associated to q. Condition (i) is thus expressed by the condition that for all $q \in X$ of codimension 1 it holds

$(i)_q$ $\quad \operatorname{mod}_q(\varphi) \le n_q$.

Using the canonical splitting of a formal group into an étale and an infinitesimal part, condition (ii) is equivalent to the condition that the following $(ii)_{\text{ét}}$ and $(ii)_{\text{inf}}$ are satisfied:

$(ii)_{\text{ét}}$ $\quad \operatorname{im}(\tau_{\varphi,\text{ét}}) \subset (\mathcal{F}_{X,D})_{\text{ét}}$,

$(ii)_{\text{inf}}$ $\quad \operatorname{im}(\tau_{\varphi,\text{inf}}) \subset (\mathcal{F}_{X,D})_{\text{inf}}$.

Let L be the affine part of G. Remember from 2.2.1 that the transformation $\tau_\varphi : L^\vee \longrightarrow \widehat{\underline{\mathrm{Div}}}_X$ is given by $\langle ?, \ell_\varphi \rangle$, where ℓ_φ is the image of $\varphi \in G(\mathcal{K}_X)$ in $\Gamma\big(G(\mathcal{K}_X)/G(\mathcal{O}_X)\big) \xrightarrow{\sim} \Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big)$, and the pairing

$$\langle ?, ? \rangle : L^\vee \times \Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(\mathbb{G}_\mathrm{m}(\mathcal{K}_X \otimes \_)/\mathbb{G}_\mathrm{m}(\mathcal{O}_X \otimes \_)\big)$$

is obtained from Cartier duality. Write $L = T \times_k U$ as a product of a torus T and a unipotent group U. Fix an isomorphism $T \cong (\mathbb{G}_\mathrm{m})^m$ and an isomorphism $U \cong (\mathbb{G}_\mathrm{a})^a$ resp. an embedding $U \subset (\mathbb{W}_r)^a$.

Let $(t_j)_{1 \le j \le m}$ be the image of ℓ_φ under

$$\Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(T(\mathcal{K}_X)/T(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(\mathbb{G}_\mathrm{m}(\mathcal{K}_X)/\mathbb{G}_\mathrm{m}(\mathcal{O}_X)\big)^m$$

and $(u_i)_{1 \le i \le a}$ be the image of ℓ_φ under

$$\Gamma\big(L(\mathcal{K}_X)/L(\mathcal{O}_X)\big) \longrightarrow \Gamma\big(U(\mathcal{K}_X)/U(\mathcal{O}_X)\big) \longrightarrow \begin{cases} \Gamma\big(\mathbb{G}_\mathrm{a}(\mathcal{K}_X)/\mathbb{G}_\mathrm{a}(\mathcal{O}_X)\big)^a \\ \Gamma\big(\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)^a. \end{cases}$$

The étale part of τ_φ is

$$\begin{array}{rccl} \tau_{\varphi,\text{ét}} : & \mathbb{Z}^m & \longrightarrow & \Gamma\big(\mathbb{G}_\mathrm{m}(\mathcal{K}_X)/\mathbb{G}_\mathrm{m}(\mathcal{O}_X)\big) \\ & (e_j)_{1 \le j \le m} & \longmapsto & \prod_{j=1}^{m} t_j^{e_j}. \end{array}$$

The image of the infinitesimal part of τ_φ is given by the image of

$$\begin{aligned}
\widetilde{\tau}_{\varphi,\mathrm{inf}} : \left.\begin{array}{l} \big(\widehat{\mathbb{G}}_{\mathrm{a}}\big)^a \\ \big({}_r\widehat{\mathbb{W}}\big)^a \end{array}\right\} &\longrightarrow \Gamma\big(\mathbb{U}_{\,?}(\mathcal{K}_X)/\mathbb{U}_{\,?}(\mathcal{O}_X)\big) \\
(v_i)_{1\le i\le a} &\longmapsto \left\{\begin{array}{l} \prod_{i=1}^a \exp(v_i\,u_i) \\ \prod_{i=1}^a \mathrm{Exp}(v_i\cdot u_i). \end{array}\right.
\end{aligned}$$

For each $q \in X$ of codimension 1 let $(t_{q,i})_{1\le j\le m}$ be a representative in $\mathbb{G}_{\mathrm{m}}(\mathcal{K}_{X,q})^m$ of the image of $(t_j)_{1\le j\le m}$ under

$$\Gamma\big(\mathbb{G}_{\mathrm{m}}(\mathcal{K}_X)/\mathbb{G}_{\mathrm{m}}(\mathcal{O}_X)\big)^m \longrightarrow \mathbb{G}_{\mathrm{m}}(\mathcal{K}_{X,q})^m/\mathbb{G}_{\mathrm{m}}(\mathcal{O}_{X,q})^m,$$

and let $(u_{q,i})_{1\le i\le a}$ be a representative in $\mathbb{G}_{\mathrm{a}}(\mathcal{K}_{X,q})^a$ resp. $\mathbb{W}_r(\mathcal{K}_{X,q})^a$ of the image of $(u_i)_{1\le i\le a}$ under

$$\Gamma\big(\mathbb{G}_{\mathrm{a}}(\mathcal{K}_X)/\mathbb{G}_{\mathrm{a}}(\mathcal{O}_X)\big)^a \longrightarrow \mathbb{G}_{\mathrm{a}}(\mathcal{K}_{X,q})^a/\mathbb{G}_{\mathrm{a}}(\mathcal{O}_{X,q})^a$$

resp.

$$\Gamma\big(\mathbb{W}_r(\mathcal{K}_X)/\mathbb{W}_r(\mathcal{O}_X)\big)^a \longrightarrow \mathbb{W}_r(\mathcal{K}_{X,q})^a/\mathbb{W}_r(\mathcal{O}_{X,q})^a.$$

Then $(ii)_{\text{ét}}$ is equivalent to the condition that

$(ii)_{\text{ét},q}$ If $n_q = 0$, then $t_{q,j} \in \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{X,q})$ for $1 \le j \le m$.

is satisfied for every point $q \in X$ of codimension 1,
whereas $(ii)_{\mathrm{inf}}$ is equivalent to the condition that

$(ii)_{\mathrm{inf},q}$ If $n_q = 0$, then $u_{q,i} \in \mathbb{G}_{\mathrm{a}}(\mathcal{O}_{X,q})$ resp. $\mathbb{W}_r(\mathcal{O}_{X,q})$ for $1 \le i \le a$.
If $n_q > 0$, then $\mathrm{n}_q(u_{q,i}) \le n_q - 1$.

is satisfied for every point $q \in X$ of codimension 1, according to Definition 3.13 of $\mathcal{F}_{X,D}$. Note that $\varphi \in G(\mathcal{O}_{X,q})$ if and only if $t_{q,j} \in \mathbb{G}_{\mathrm{m}}(\mathcal{O}_{X,q})$ for $1 \le j \le m$ and $u_{q,i} \in \mathbb{G}_{\mathrm{a}}(\mathcal{O}_{X,q})$ resp. $\mathbb{W}_r(\mathcal{O}_{X,q})$ for $1 \le i \le a$. By Definition 3.11, for each $q \in X$ of codimension 1

$(i)_q$ $\mathrm{mod}_q(\varphi) \le n_q$

if and only if $(ii)_{\text{ét},q}$ and $(ii)_{\mathrm{inf},q}$ are satisfied. ■

Theorem 3.18. *The category* $\mathbf{Mr}^{X,D}$ *of rational maps of modulus* $\le D$ *is equivalent to the category* $\mathbf{Mr}_{\mathcal{F}_{X,D}}$ *of rational maps which induce a transformation to* $\mathcal{F}_{X,D}$*.*

Proof. According to the definitions of $\mathbf{Mr}^{X,D}$ and $\mathbf{Mr}_{\mathcal{F}_{X,D}}$, the statement is due to Lemma 3.17. ■

Theorem 3.19. *The Albanese* $\mathrm{Alb}(X,D)$ *of* X *of modulus* D *exists and is dual (in the sense of 1-motives) to the 1-motive* $\big[\mathcal{F}_{X,D}^{0,\mathrm{red}} \longrightarrow \mathrm{Pic}_X^{0,\mathrm{red}}\big]$.

Proof. By Theorem 3.18, $\mathrm{Alb}(X,D)$ is the universal object of the category $\mathbf{Mr}_{\mathcal{F}_{X,D}}$ (if it exists). A rational map from X to an algebraic group induces a transformation to $\mathcal{F}_{X,D}$ if and only if it induces a transformation to $\mathcal{F}_{X,D}^{0,\mathrm{red}}$, by Lemma 2.6. Since $\mathcal{F}_{X,D}^{0,\mathrm{red}}$ is dual-algebraic (Proposition 3.14), the category $\mathbf{Mr}_{\mathcal{F}_{X,D}}$ admits a universal object (Theorem 2.12), and this universal object is dual to $\left[\mathcal{F}_{X,D}^{0,\mathrm{red}} \longrightarrow \mathrm{Pic}_X^{0,\mathrm{red}}\right]$ (Remark 2.14). ■

Corollary 3.20. *For every rational map φ from X to a smooth connected algebraic group G there exists an effective divisor D, namely $D = \mathrm{mod}\,(\varphi)$, such that φ factors through $\mathrm{Alb}(X,D)$.*

Proposition 3.21. *Let $\mathcal{F}$ be a formal subgroup of $\underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. Then $\mathcal{F}$ is dual-algebraic if and only if there exists an effective divisor D such that $\mathcal{F} \subset \mathcal{F}_{X,D}$.*

Proof. ($\Longleftarrow$) A formal subgroup of a dual-algebraic group is also dual-algebraic, according to Lemma 3.15.

($\Longrightarrow$) Let $D = \mathrm{mod}\,(\mathrm{alb}_{\mathcal{F}})$ be the modulus of the universal rational map $\mathrm{alb}_{\mathcal{F}} : X \longrightarrow \mathrm{Alb}_{\mathcal{F}}(X)$ associated to $\mathcal{F} \subset \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$. Then by Lemma 3.17 it holds $\mathcal{F} = \mathrm{im}\,(\tau_{\mathrm{alb}_{\mathcal{F}}}) \subset \mathcal{F}_{X,D}$. ■

3.2.2 Functoriality

We specialize the results from No. 2.3.2 to the case of Albanese varieties with modulus.

Proposition 3.22. *Let $\psi : Y \longrightarrow X$ be a morphism of smooth proper varieties. Let D be an effective divisor on X intersecting $\psi(Y)$ properly. Then ψ induces a homomorphism of algebraic groups*

$$\mathrm{Alb}_{Y,E}^{X,D}(\psi) : \mathrm{Alb}(Y,E) \longrightarrow \mathrm{Alb}(X,D)$$

for each effective divisor E on Y satisfying $E \geq (D - D_{\mathrm{red}}) \cdot Y + (D \cdot Y)_{\mathrm{red}}$.

Proof. According to Proposition 2.15, for the existence of $\mathrm{Alb}_{Y,E}^{X,D}(\psi)$ it is sufficient to show $\mathcal{F}_{Y,E} \supset \mathcal{F}_{X,D} \cdot Y$. Definition 3.13 of $\mathcal{F}_{X,D}$ implies that this is the case if and only if $\mathrm{Supp}(E) \supset \mathrm{Supp}(D \cdot Y)$ and $E - E_{\mathrm{red}} \geq (D - D_{\mathrm{red}}) \cdot Y$. But this is equivalent to $E \geq (D - D_{\mathrm{red}}) \cdot Y + (D \cdot Y)_{\mathrm{red}}$. ■

Corollary 3.23. *If D and E are effective divisors on X with $E \geq D$, then there is a canonical surjective homomorphism*

$$\mathrm{Alb}(X,E) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Alb}(X,D)$$

given by $\mathrm{Alb}_{X,E}^{X,D}(\mathrm{id}_X)$.

Proof. If $E \geq D$, then it is evident that $\mathrm{Alb}(X,E)$ generates $\mathrm{Alb}(X,D)$, thus $\mathrm{Alb}_{X,E}^{X,D}(\mathrm{id}_X)$ is surjective. ■

3.2.3 Descent of the Base Field

Let k be a perfect field. Let $\overline{k}$ be an algebraic closure of k. Let X be a smooth proper variety defined over k, and let D be an effective divisor on X rational over k.

Theorem 3.24. *There exists a k-torsor $\mathrm{Alb}^{(1)}(X,D)$ for an algebraic k-group $\mathrm{Alb}^{(0)}(X,D)$ and rational maps defined over k*

$$\mathrm{alb}_{X,D}^{(i)} : X^{2-i} \dashrightarrow \mathrm{Alb}^{(i)}(X,D)$$

for $i = 1, 0$, satisfying the following universal property:

If $\varphi : X \dashrightarrow G^{(1)}$ is a rational map defined over k to a k-torsor $G^{(1)}$ for an algebraic k-group $G^{(0)}$, such that $\operatorname{mod}\big(\varphi \otimes_k \overline{k}\big) \leq D \otimes_k \overline{k}$, then there exist a unique affine homomorphism of k-torsors $h^{(1)}$ and a unique homomorphism of algebraic k-groups $h^{(0)}$, $h^{(i)} : \mathrm{Alb}^{(i)}(X,D) \longrightarrow G^{(i)}$, defined over k, such that $\varphi^{(i)} = h^{(i)} \circ \mathrm{alb}_{X,D}^{(i)}$ for $i = 1, 0$.

Proof. Follows directly from Theorem 2.20. ■

3.3 Jacobian with Modulus

Let C be a smooth proper curve over k. Let $D = \sum_{q \in S} n_q\, q$ be an effective divisor on C, where S is a finite set of closed points on C and n_q are integers ≥ 1 for $q \in S$. The Jacobian $J(C,D)$ of C of modulus D is by definition the universal object for the category of those morphisms φ from $C \setminus S$ to algebraic groups such that $\varphi\big(\operatorname{div}(f)\big) = 0$ for all $f \in \mathcal{K}_C$ with $f \equiv 1 \mod D$. Here we used the definition $\varphi\big(\sum l_j c_j\big) = \sum l_j\, \varphi(c_j)$ for a divisor $\sum l_j c_j$ on C with $c_j \in C \setminus S$, and "$f \equiv 1 \mod D$" means $\mathrm{v}_q(1-f) \geq n_q$ for all $q \in S$, where v_q is the valuation attached to the point $q \in C$.

Theorem 3.25. *The generalized Jacobian $J(C,D)$ of C of modulus D is an extension*

$$0 \longrightarrow L(C,D) \longrightarrow J(C,D) \longrightarrow J(C) \longrightarrow 0$$

of the classical Jacobian $J(C) \cong \mathrm{Pic}^0_C$ of C, which is an abelian variety, by the affine algebraic group

$$L(C,D) = \frac{\prod_{q\in S} \mathbb{G}_{\mathrm{m},q}}{\mathbb{G}_{\mathrm{m}}} \times \prod_{q \in S} \mathbb{U}_{\mathcal{O}_{C,q}/\mathfrak{m}_{C,q}^{n_q}}$$

where $\mathbb{G}_{\mathrm{m},q}$ denotes a group isomorphic to the multiplicative group and attached to the point q, where $\mathbb{U}_R$ is the unipotent group associated to a finite k-algebra R from 1.1.6, and $(\mathcal{O}_{C,q}, \mathfrak{m}_{C,q})$ is the local ring at $q \in S$.

Proof. [Ser3, V, § 3]. ∎

We give an illustration of Theorem 3.25, cf. [Ser3, I, No. 1]. The classical Jacobian $J(C)$ is the group of divisor classes on C of degree 0. The Jacobian $J(C,D)$ of modulus D is identified to the group of classes of divisors prime to S modulo principal divisors $\operatorname{div}(f)$ with $f \equiv 1 \mod D$. Then the affine part $L(C,D)$ of $J(C,D)$, as the kernel of the canonical surjection $J(C,D) \twoheadrightarrow J(C)$, is characterized by

$$\begin{aligned}
L(C,D)(k) &= \frac{\left\{\operatorname{div}(f) \mid f \in \mathcal{O}_{C,q}^* \quad \forall q \in S\right\}}{\{\operatorname{div}(f) \mid \mathrm{v}_q(1-f) \geq n_q \quad \forall q \in S\}} \\
&= \frac{\left\{f \in \mathcal{K}_C \mid f \in \mathcal{O}_{C,q}^* \quad \forall q \in S\right\}}{k^* \times \{f \in \mathcal{K}_C \mid \mathrm{v}_q(1-f) \geq n_q \quad \forall q \in S\}} \\
&= \frac{\prod_{q\in S} \mathcal{O}_{C,q}^*}{k^* \times \prod_{q \in S}\left(1+\mathfrak{m}_q^{n_q}\right)} \\
&= \frac{\prod_{q\in S} k(q)^*}{k^*} \times \prod_{q\in S} \frac{1+\mathfrak{m}_q}{1+\mathfrak{m}_q^{n_q}}
\end{aligned}$$

where $k(q)$ denotes the residue field and $\mathfrak{m}_q$ the maximal ideal at $q \in C$.

Theorem 3.26. *The Jacobian with modulus $J(C,D)$ is dual (in the sense of 1-motives) to the 1-motive $\left[\mathcal{F}_{C,D}^0 \longrightarrow \operatorname{Pic}_C^0\right]$, where $\mathcal{F}_{C,D}^0 = \mathcal{F}_{C,D}^{0,\mathrm{red}}$ is the formal subgroup of $\underline{\operatorname{Div}}_C^0$ from Definition 3.13, and $\mathcal{F}_{C,D}^0 \longrightarrow \operatorname{Pic}_C^0$ is the homomorphism induced by the class map $\underline{\operatorname{Div}}_C^0 \longrightarrow \underline{\operatorname{Pic}}_C^0$.*

Proof. We have to ensure that the category for which $J(C,D)$ is universal is characterized by the formal group $\mathcal{F}_{C,D}$. The Jacobian $J(C,D)$ of modulus D is by definition the universal object for morphisms φ from $C \setminus S$ to algebraic groups satisfying

(i) $\varphi\big(\operatorname{div}(f)\big) = 0 \qquad \forall f \in \mathcal{K}_C$ with $f \equiv 1 \mod D$.

Condition (i) is equivalent to

(ii) $(\varphi, f)_q = 0 \qquad \forall q \in S, \forall f \in \mathcal{K}_C$ with $f \equiv 1 \mod D$ at q

where $(\varphi, ?)_? : \mathcal{K}_C^* \times C \longrightarrow G(k)$ is the local symbol associated to the morphism $\varphi : C \setminus S \longrightarrow G$, according to [Ser3, I, No. 1, Thm. 1 and III, § 1]. It is shown in [KR2, No. 6.1-3] that condition (ii) is equivalent to

(iii) $\operatorname{mod}(\varphi) \leq D$.

Then the assertion is due to Theorems 3.18 and 3.19. ∎

3.4 Relative Chow Group with Modulus

Let X be a smooth proper variety over an algebraically closed field k, let D be an effective divisor on X and D_{red} the reduced part of D.

Notation 3.27. If C is a curve in X, then $\nu : \widetilde{C} \longrightarrow C$ denotes the normalization. For $f \in \mathcal{K}_C$, we write $\widetilde{f} := \nu^* f$ for the image of f in $\mathcal{K}_{\widetilde{C}}$. If $\varphi : X \dashrightarrow G$ is a rational map, we write $\varphi|_{\widetilde{C}} := \varphi|_C \circ \nu$ for the composition of φ and ν. If B is a Cartier divisor on X intersecting C properly, then $B \cdot \widetilde{C}$ denotes the pull-back of B to $\widetilde{C}$.

Definition 3.28. Let $\mathrm{Z}_0\,(X \setminus D)$ be the group of 0-cycles on $X \setminus D$, set

$$\mathcal{R}_0(X,D) = \left\{ (C,f) \,\middle|\, \begin{array}{l} C \text{ a curve in } X \text{ intersecting } \operatorname{Supp}(D) \text{ properly,} \\ f \in \mathcal{K}_C^* \text{ s.t. } \widetilde{f} \equiv 1 \mod (D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}} \end{array} \right\}$$

and let $\mathrm{R}_0\,(X,D)$ be the subgroup of $\mathrm{Z}_0\,(X \setminus D)$ generated by the elements $\operatorname{div}(f)_C$ with $(C,f) \in \mathcal{R}_0(X,D)$. Then define

$$\mathrm{CH}_0(X,D) = \mathrm{Z}_0(X \setminus D) \big/ \mathrm{R}_0(X,D).$$

Let $\mathrm{CH}_0(X,D)^0$ be the subgroup of $\mathrm{CH}_0(X,D)$ of cycles ζ with $\deg \zeta|_W = 0$ for all irreducible components W of $X \setminus D$.

Definition 3.29. Let $\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$ be the category of rational maps from X to algebraic groups defined as follows: the objects of $\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$ are morphisms $\varphi : X \setminus D \longrightarrow G$ whose associated map on 0-cycles of degree 0

$$\begin{array}{rcl} \mathrm{Z}_0(X \setminus D)^0 & \longrightarrow & G(k) \\ \sum l_i\, p_i & \longmapsto & \sum l_i\, \varphi(p_i) \end{array}$$

factors through a homomorphism of groups $\mathrm{CH}_0(X,D)^0 \longrightarrow G(k)$. [2]
We refer to the objects of $\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$ as rational maps from X to algebraic groups *factoring through* $\mathrm{CH}_0(X,D)^0$.

Theorem 3.30. *The category* $\mathbf{Mr}^{X,D}$ *of rational maps of modulus* $\leq D$ *is equivalent to the category* $\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$ *of rational maps factoring through* $\mathrm{CH}_0(X,D)^0$*. In particular, the Albanese* $\mathrm{Alb}(X,D)$ *of* X *of modulus* D *is the universal quotient of* $\mathrm{CH}_0(X,D)^0$.

[2] A category of rational maps to algebraic groups is defined already by its objects.

Proof. According to the definitions of $\mathbf{Mr}^{X,D}$ and $\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$ the task is to show that for a morphism $\varphi : X \setminus D \longrightarrow G$ from $X \setminus D$ to a smooth connected algebraic group G the following conditions are equivalent:

(i) $\operatorname{mod}(\varphi) \leq D$,

(ii) $\varphi(\operatorname{div}(f)_C) = 0 \qquad \forall\ (C, f) \in \mathcal{R}_0(X, D)$.

Since $\varphi(\operatorname{div}(f)_C) = \varphi|_{\widetilde{C}}\big(\operatorname{div}\big(\widetilde{f}\big)_{\widetilde{C}}\big)$ (see [Ru, Lemma 3.32]), condition (ii) is equivalent to the condition

(ii') $\operatorname{mod}\big(\varphi|_{\widetilde{C}}\big) \leq (D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}}$
for all curves C in X intersecting $\operatorname{Supp}(D)$ properly,

as was seen in the proof of Theorem 3.26, substituting D by $(D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}}$. The equivalence of (i) and (ii') is the content of Lemma 3.31. ■

Lemma 3.31. *Let $\varphi : X \dashrightarrow G$ be a rational map from X to a smooth connected algebraic group G. Then the following conditions are equivalent:*

(i) $\operatorname{mod}(\varphi) \leq D$,

(ii) $\operatorname{mod}\big(\varphi|_{\widetilde{C}}\big) \leq (D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}}$
for all curves C in X intersecting $\operatorname{Supp}(D)$ *properly.*

Proof. (i)$\Longrightarrow$(ii) Let C be a curve in X intersecting D properly. As φ is regular away from D, the restriction $\varphi|_{\widetilde{C}}$ of φ to $\widetilde{C}$ is regular away from $D \cdot \widetilde{C}$. Hence $\operatorname{Supp}\big(\operatorname{mod}\big(\varphi|_{\widetilde{C}}\big)\big) \subset \operatorname{Supp}\big(D \cdot \widetilde{C}\big) = \operatorname{Supp}\big((D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}}\big)$. According to Definition 3.11 of the modulus, it is easy to see that $\operatorname{mod}(\varphi) \leq D = (D - D_{\mathrm{red}}) + D_{\mathrm{red}}$ implies $\operatorname{mod}\big(\varphi|_{\widetilde{C}}\big) \leq (D - D_{\mathrm{red}}) \cdot \widetilde{C} + \big(D \cdot \widetilde{C}\big)_{\mathrm{red}}$.

(ii)$\Longrightarrow$(i) Let $E := \operatorname{mod}(\varphi)$ and $q \in \operatorname{Supp}(E)$ be a point of codimension 1 in X. We are going to construct a family of smooth curves $\{C_e\}_e$ intersecting E in a fixed point $x \in E_q = \overline{\{q\}}$ such that

$$\lim_{e \to \infty} \frac{\operatorname{mod}_x\left(\varphi|_{C_e}\right)}{\mu_x\big((E - E_{\mathrm{red}}) \cdot C_e\big) + 1} = 1$$

where $\mu_x(E \cdot C)$ denotes the intersection multiplicity of E and C at x.

After the construction we will show that the existence of such a family of curves for each $q \in \operatorname{Supp}(E)$ of codimension 1 in X yields the implication (ii)$\Longrightarrow$(i).

If $\operatorname{char}(k) = 0$, it is easy to see that a general curve C in X intersecting E_q in a point x satisfies $\operatorname{mod}_x(\varphi|_C) = \mu_x\big((E - E_{\mathrm{red}}) \cdot C\big) + 1$. Therefore we suppose that $\operatorname{char}(k) = p > 0$. Using the notation of Definition 3.11, let $(u_{q,i})_{1 \leq i \leq a} \in \mathbb{W}_r(\mathcal{K}_{X,q})^a$ be a representative of the unipotent part of the class of $\varphi \in G(\mathcal{K}_{X,q})$ in $G(\mathcal{K}_{X,q})/G(\mathcal{O}_{X,q}) = L(\mathcal{K}_{X,q})/L(\mathcal{O}_{X,q})$. Then $\operatorname{mod}_q(\varphi) = 1 + \mathrm{n}_q(u_{q,i})$ for some $1 \leq i \leq a$. Set $n := \mathrm{n}_q(u_{q,i})$. Let $t \in \mathfrak{m}_{X,q}$ be

a uniformizer at q. Let $\sum_\nu \mathrm{F}^\nu \otimes \omega_\nu \otimes t^{-n} \in k[\mathrm{F}] \otimes_k \Omega_{X,q}(\log q) \otimes_{\mathcal{O}_{X,q}} \mathfrak{m}_{X,q}^{-n}$ be a representative of $\overline{\mathfrak{d}}_{nq}(u_{q,i}) \in \overline{\mathfrak{D}}_{nq}$ (Definition 3.7). Choose a regular closed point $x \in E_q$ such that t is a local equation for E_q at x and ω_ν is regular and $\neq 0$ at x for some ν. We may assume that $\dim X = 2$ via cutting down by hyperplanes through x transversal to E_q. Let $s \in \mathfrak{m}_{X,x}$ be a local parameter at x that gives a uniformizer of $\mathcal{O}_{E_q,x}$. Define a curve C_e locally around x by the equation $t = s^e$ for $e \geq 1$. Note that $E - E_{\mathrm{red}}$ is locally defined by the equation $t^n = 0$. Then

$$\mu_x\big((E - E_{\mathrm{red}}) \cdot C_e\big) = \dim_k \frac{\mathcal{O}_{X,x}}{(t^n, t - s^e)} = ne.$$

We can write $\omega_\nu = g\,\mathrm{d}s + h\,\mathrm{d}\log t$ with $g, h \in \mathcal{O}_{X,q}$ and the values at x are $g(x) \neq 0$ if $\overline{\mathfrak{d}}_{nq}(u_{q,i}) \in {}^\flat\overline{\mathfrak{D}}_{nq}$, $h(x) \neq 0$ if $\overline{\mathfrak{d}}_{nq}(u_{q,i}) \in \overline{\mathfrak{D}}_{nq} \setminus {}^\flat\overline{\mathfrak{D}}_{nq}$ and x in general position (what we assume), for some ν. The restriction of $t^{-n}\omega_\nu$ to C_e is

$$\begin{aligned} t^{-n}\omega_\nu|_{C_e} &= s^{-ne} g\,\mathrm{d}s + s^{-ne} h\,\mathrm{d}\log s^e \\ &= s^{1-ne} g\,\mathrm{d}\log s + e\, s^{-ne} h\,\mathrm{d}\log s, \end{aligned}$$

and the class of $t^{-n}\omega_\nu|_{C_e}$ is non-zero in

$$\begin{cases} \Omega_{C_e,x}(\log x) \otimes_{\mathcal{O}_{C_e,x}} \mathfrak{m}_{C_e,x}^{-ne}/\mathfrak{m}_{C_e,x}^{1-ne} & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in \overline{\mathfrak{D}}_{nq} \setminus {}^\flat\overline{\mathfrak{D}}_{nq} \text{ and } p \nmid e \\ \Omega_{C_e,x}(\log x) \otimes_{\mathcal{O}_{C_e,x}} \mathfrak{m}_{C_e,x}^{1-ne}/\mathfrak{m}_{C_e,x}^{2-ne} & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in {}^\flat\overline{\mathfrak{D}}_{nq}. \end{cases}$$

Lemma 3.10 assures that the modulus of $\varphi|_{C_e}$ is computed from the restriction (of a representative) of $\overline{\mathfrak{d}}_{nq}(u_{q,i})$ to C_e, for e large enough such that $ne - 1 > \lfloor ne/p \rfloor$ (this is satisfied for $e > 2$). Thus we have

$$\mathrm{n}_x\left(u_{q,i}|_{C_e}\right) = \begin{cases} ne & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in \overline{\mathfrak{D}}_{nq} \setminus {}^\flat\overline{\mathfrak{D}}_{nq} \text{ and } p \nmid e \\ ne - 1 & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in {}^\flat\overline{\mathfrak{D}}_{nq}, \end{cases}$$

$$\mathrm{mod}_x\left(\varphi|_{C_e}\right) = \begin{cases} ne + 1 & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in \overline{\mathfrak{D}}_{nq} \setminus {}^\flat\overline{\mathfrak{D}}_{nq} \text{ and } p \nmid e \\ ne & \text{if } \overline{\mathfrak{d}}_{nq}(u_{q,i}) \in {}^\flat\overline{\mathfrak{D}}_{nq}. \end{cases}$$

Then

$$\lim_{e \to \infty} \frac{\mathrm{mod}_x\left(\varphi|_{C_e}\right)}{\mu_x\big((E - E_{\mathrm{red}}) \cdot C_e\big) + 1} = 1.$$

Now we show $\neg(i) \Longrightarrow \neg(ii)$. Suppose $E := \mathrm{mod}\,(\varphi) \not\leq D$. Then there is a point $q \in \mathrm{Supp}(E)$ of codimension 1 in X such that $\mu_q(E) > \mu_q(D)$, where μ_q is the multiplicity at q. By the construction above there is a sequence

of curves $\{C_e\}_e$ in X intersecting E in a fixed point $x \in E_q$ such that $\lim\limits_{e\to\infty} \frac{\mathrm{mod}_x(\varphi|_{C_e})}{\mu_x((E-E_{\mathrm{red}})\cdot C_e)+1} = 1$. If $\mu_q(D) \neq 0$, then since $\sup\limits_{e\geq 0} \frac{\mu_x((D-D_{\mathrm{red}})\cdot C_e)+1}{\mu_x((E-E_{\mathrm{red}})\cdot C_e)+1} < 1$ there is e such that $\mathrm{mod}_x(\varphi|_{C_e}) > \mu_x\big((D-D_{\mathrm{red}})\cdot C_e\big)+1$. If $\mu_q(D)=0$, then $0 \neq \mathrm{mod}(\varphi|_{C_e})_x > \mu_x\big((D-D_{\mathrm{red}})\cdot C_e + (D\cdot C_e)_{\mathrm{red}}\big) = 0$. Thus $\mathrm{mod}(\varphi|_{C_e}) \not\leq (D-D_{\mathrm{red}})\cdot C_e + (D\cdot C_e)_{\mathrm{red}}$. ■

3.5 Class Field Theory

Let X be a geometrically irreducible smooth proper variety over $k = \mathbb{F}_q$, a finite field with q elements. Let $\overline{k}$ be an algebraic closure of k, we denote $\overline{X} = X \otimes_k \overline{k}$. We write K_X for the function field of X. The aim of this subsection is to determine the Galois group $\mathrm{Gal}(\mathrm{K}_X^{\mathrm{ab}}/\mathrm{K}_X)$ of the maximal abelian extension $\mathrm{K}_X^{\mathrm{ab}}$ of K_X.

There is an exact sequence

$$1 \longrightarrow \mathrm{Gal}\big(\mathrm{K}_X^{\mathrm{ab}}/\mathrm{K}_{\overline{X}}\big) \longrightarrow \mathrm{Gal}\big(\mathrm{K}_X^{\mathrm{ab}}/\mathrm{K}_X\big) \longrightarrow \mathrm{Gal}\big(\mathrm{K}_{\overline{X}}/\mathrm{K}_X\big) \longrightarrow 1$$

where $\mathrm{Gal}\big(\mathrm{K}_{\overline{X}}/\mathrm{K}_X\big) = \mathrm{Gal}\big(\mathrm{K}_X\,\overline{k}/\mathrm{K}_X\big) = \mathrm{Gal}\big(\overline{k}/k\big) = \widehat{\mathbb{Z}}$.

Definition 3.32. We write $\widehat{\mathfrak{g}}(\mathrm{K}_X) = \mathrm{Gal}\big(\mathrm{K}_X^{\mathrm{ab}}/\mathrm{K}_X\big)$ for the Galois group of the maximal abelian extension of K_X, and $\mathfrak{g}^0(\mathrm{K}_X) = \mathrm{Gal}\big(\mathrm{K}_X^{\mathrm{ab}}/\mathrm{K}_{\overline{X}}\big)$ for the geometric Galois group. We denote by $\mathfrak{g}(\mathrm{K}_X)$ the inverse image of $\mathbb{Z}$ in $\widehat{\mathfrak{g}}(\mathrm{K}_X)$ under $\widehat{\mathfrak{g}}(\mathrm{K}_X) \longrightarrow \widehat{\mathbb{Z}}$. Then $\widehat{\mathfrak{g}}(\mathrm{K}_X)$ is the completion of $\mathfrak{g}(\mathrm{K}_X)$ for the topology defined by the subgroups of finite index.

Proposition 3.33. *For $k=\mathbb{F}_q$, every k-torsor P for an algebraic k-group G admits a k-rational point p. Hence there is an identification $G \xrightarrow{\sim} P$ given by $g \longmapsto g\cdot p$.*

Proof. [Ser3, VI, No. 4, Cor. 1 of Prop. 3]. ■

Definition 3.34. Let P be a k-torsor for a k-group G. Let

$$\mathcal{Z}_k(P) = \left\{ \sum_i l_i\,p_i \;\middle|\; l_i \in \mathbb{Z},\ p_i \in P(k) \right\}$$

be the free abelian group generated by the k-rational points of P. This group admits a surjective homomorphism $\deg : \mathcal{Z}_k(P) \longrightarrow \mathbb{Z}$. Let

$$\mathcal{Z}_k(P)^0 = \ker\Big(\deg : \mathcal{Z}_k(P) \longrightarrow \mathbb{Z}\Big)$$

be the kernel of deg. Each element $z \in \mathcal{Z}_k(P)^0$ is a formal sum $z = \sum_i l_i\, p_i$ with $\sum_i l_i = 0$. As P is a k-torsor for the k-group G, one can associate to such an element z the same sum, computed in the group $G(k)$ (cf. Notation 2.17). This yields a surjective homomorphism $\mathcal{Z}_k(P)^0 \longrightarrow G(k)$. Let

$$\mathcal{N}_k(P) \;=\; \ker\Big(\mathcal{Z}_k(P)^0 \longrightarrow G(k)\Big)$$

be the kernel of this homomorphism.

The homomorphism obtained from $\deg : \mathcal{Z}_k(P) \longrightarrow \mathbb{Z}$ by passage to the quotient $\mathcal{Z}_k(P)\big/\mathcal{N}_k(P)$ admits as kernel $\mathcal{Z}_k(P)^0\big/\mathcal{N}_k(P) \cong G(k)$. Thus we have an exact sequence

$$0 \longrightarrow G(k) \longrightarrow \frac{\mathcal{Z}_k(P)}{\mathcal{N}_k(P)} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

By the results of No. 2.3.3, for each effective divisor D on X rational over k there is a universal rational map $\mathrm{alb}^{(1)}_{X,D} : X \dashrightarrow \mathrm{Alb}^{(1)}(X,D)$ defined over k from X to a k-torsor $\mathrm{Alb}^{(1)}(X,D)$ for an algebraic k-group $\mathrm{Alb}^{(0)}(X,D)$. For $E \geq D$ the universal properties of $\mathrm{Alb}^{(i)}(X,E)$ yield surjective homomorphisms $\mathrm{Alb}^{(i)}(X,E) \twoheadrightarrow \mathrm{Alb}^{(i)}(X,D)$ for $i = 1, 0$. Thus the inductive system {effective divisors D rational over k} gives rise to a projective system $\big\{\, \mathrm{Alb}^{(i)}(X,D)\big\}_D$ for $i = 1, 0$. The exact sequence from above is compatible with the associated maps for $E \geq D$, i.e. we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Alb}^{(0)}(X,E)(k) \arrow[r] \arrow[d, two heads] & \frac{\mathcal{Z}_k(\mathrm{Alb}^{(1)}(X,E))}{\mathcal{N}_k(\mathrm{Alb}^{(1)}(X,E))} \arrow[r] \arrow[d, two heads] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Alb}^{(0)}(X,D)(k) \arrow[r] & \frac{\mathcal{Z}_k(\mathrm{Alb}^{(1)}(X,D))}{\mathcal{N}_k(\mathrm{Alb}^{(1)}(X,D))} \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

This allows the following

Definition 3.35.

$$\mathcal{A}_k(X)^0 \;=\; \varprojlim_D \mathrm{Alb}^{(0)}(X,D)(k)$$

$$\mathcal{A}_k(X) \;=\; \varprojlim_D \frac{\mathcal{Z}_k\big(\,\mathrm{Alb}^{(1)}(X,D)\big)}{\mathcal{N}_k\big(\,\mathrm{Alb}^{(1)}(X,D)\big)}$$

where D ranges over all effective divisors D on X rational over k.

Theorem 3.36. *There exists a canonical isomorphism of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}^0\left(\mathrm{K}_X\right) \arrow[r] \arrow[d, "\wr"] & \mathfrak{g}\left(\mathrm{K}_X\right) \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}_k(X)^0 \arrow[r] & \mathcal{A}_k(X) \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Proof. The proof is analogous to the proof of Lang's class field theory given in [Ser3, VI, § 4, No. 16-19], replacing *maximal maps* by the universal maps $\mathrm{alb}^{(1)}_{X,D} : X \dashrightarrow \mathrm{Alb}^{(1)}(X,D)$. ∎

We review the proof of $\mathfrak{g}^0\left(\mathrm{K}_X\right) \cong \mathcal{A}_k(X)^0$.
Let $\overline{Y} \dashrightarrow \overline{X}$ a finite abelian covering (in the birational sense) corresponding to a finite abelian extension $\mathrm{K}_{\overline{Y}} / \mathrm{K}_{\overline{X}}$ in $\mathrm{K}_X^{\mathrm{ab}}$.

Proposition 3.37. *Every abelian covering is the pull-back of a separable isogeny.*

Proof. [Ser3, Chapter VI, No. 8, Corollary of Proposition 7]. ∎

Thus there exists a rational map $\varphi : \overline{X} \dashrightarrow G$ from $\overline{X}$ to an algebraic group G and an isogeny of algebraic groups $H \to G$ such that $\overline{Y} = \varphi^* H$, i.e. the following diagram is a fibre-square

$$\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d] & H \arrow[d] \\
\overline{X} \arrow[r, "\varphi"] & G.
\end{tikzcd}$$

For $D \geq \mathrm{mod}\,(\varphi)$, the universal property of $\mathrm{Alb}(\overline{X}, D)$ yields a homomorphism $h : \mathrm{Alb}(\overline{X}, D) \longrightarrow G$ such that $\varphi = h \circ \mathrm{alb}_{\overline{X},D}$. Hence $\overline{Y} \dashrightarrow \overline{X}$ is the pull-back of an isogeny over $\mathrm{Alb}(\overline{X}, D)$. Enlarging D if necessary, we may assume that D is defined over k: Let k_1/k be a finite extension of k such that k_1 is a field of definition for D. Then $D' = \sum_\sigma D^\sigma$ is defined over k, where D^σ are the conjugates of D by means of $\sigma \in \mathrm{Gal}(k_1/k)$. As $\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_X\right) \cong \mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_{\overline{X}}\right) \times \mathrm{Gal}\left(\overline{k}/k\right)$, the subgroup of finite index $\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_{\overline{Y}}\right) \subset \mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_{\overline{X}}\right)$ yields the subgroup $\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_{\overline{Y}}\right) \times \mathrm{Gal}\left(\overline{k}/k\right) \cong \mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_Y\right) \subset \mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} / \mathrm{K}_X\right)$ of finite index, which corresponds to a finite abelian extension $\mathrm{K}_Y / \mathrm{K}_X$. This determines a finite abelian covering $Y \dashrightarrow X$ over k, which is the pull-back of an isogeny over $\mathrm{Alb}^{(1)}(X, D)$. According to Proposition 3.33 we may

identify $\mathrm{Alb}^{(1)}(X,D) \cong \mathrm{Alb}^{(0)}(X,D)$ and denote the universal map just by $\mathrm{alb}_{X,D} : X \dashrightarrow \mathrm{Alb}(X,D)$.

Proposition 3.38. *Let $i : H \longrightarrow G$ be a separable isogeny defined over $k = \mathbb{F}_q$. Then the following conditions are equivalent:*

(i) *The extension $\mathrm{K}_H \,/\, \mathrm{K}_G$ defined by i is galois.*

(ii) *The extension $\mathrm{K}_H \,/\, \mathrm{K}_G$ defined by i is abelian.*

(iii) $\ker(i) \subset G(k) =$ *k-rational points of G.*

(iv) *i is a quotient of $\mathrm{F} - \mathrm{id}$ (Frobenius minus identity), i.e. there is a homomorphism $h : G \longrightarrow H$ such that $\mathrm{F} - \mathrm{id} : G \longrightarrow G$ factors as $G \xrightarrow{h} H \xrightarrow{i} G$.*

Proof. [Ser3, Chapter VI, No. 6, Proposition 6]. ■

Then $Y \dashrightarrow X$ is a quotient of $\mathrm{alb}^*_{X,D}\,\mathrm{Alb}(X,D) \longrightarrow X$, i.e. we have the following fibre-square

$$\begin{tikzcd}
\mathrm{alb}^*_{X,D}\,\mathrm{Alb}(X,D) \arrow[r] \arrow[d] & \mathrm{Alb}(X,D) \arrow[dd, "\mathrm{F}-\mathrm{id}"] \\
Y \arrow[d] & \\
X \arrow[r, "\mathrm{alb}_{X,D}"] & \mathrm{Alb}(X,D).
\end{tikzcd}$$

The kernel of $\mathrm{F} - \mathrm{id} : \mathrm{Alb}(X,D) \longrightarrow \mathrm{Alb}(X,D)$ is given by the k-rational points of $\mathrm{Alb}(X,D)$.

$$0 \longrightarrow \mathrm{Alb}(X,D)(k) \longrightarrow \mathrm{Alb}(X,D) \xrightarrow{\mathrm{F}-\mathrm{id}} \mathrm{Alb}(X,D) \longrightarrow 0.$$

The Galois group of the extension is isomorphic to the fibre of the covering:

$$\mathrm{Gal}\left(\mathrm{K}_{\mathrm{alb}^*_{X,D}\,\mathrm{Alb}(X,D)} \Big/ \mathrm{K}_X\right) \cong \mathrm{Alb}(X,D)(k).$$

Hence the Galois group $\mathrm{Gal}(\mathrm{K}_Y \,/\, \mathrm{K}_X)$ is a quotient of $\mathrm{Alb}(X,D)(k)$.

Now $\mathrm{Gal}(\mathrm{K}_X^{\mathrm{ab}} \,/\, \mathrm{K}_{\overline{X}})$ is the projective limit of Galois groups $\mathrm{Gal}(\mathrm{K}_{\overline{Y}} \,/\, \mathrm{K}_{\overline{X}}) \cong \mathrm{Gal}(\mathrm{K}_Y \,/\, \mathrm{K}_X)$ as above. Thus $\mathrm{Gal}(\mathrm{K}_X^{\mathrm{ab}} \,/\, \mathrm{K}_{\overline{X}})$ is given by the projective limit of $\mathrm{Alb}(X,D)(k)$ ranging over all effective divisors D on X rational over k:

$$\mathrm{Gal}\left(\mathrm{K}_X^{\mathrm{ab}} \,/\, \mathrm{K}_{\overline{X}}\right) \cong \varprojlim_{D} \mathrm{Alb}(X,D)(k).$$

Glossary

Categories

Set	sets	1.1.1
Ab	abelian groups	1.1.1
$\mathbf{Alg}/k$	k-algebras	1.1.1
$\mathbf{Art}/k$	k-algebras of finite length	1.1.1
$\mathbf{Fctr}(\mathfrak{A}, \mathfrak{B})$	functors from $\mathfrak{A}$ to $\mathfrak{B}$	1.1.1
$\mathbf{Ab}/k$	k-group functors $(= \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab}))$	1.1.1
$\mathcal{A}b/k$	k-group sheaves (for fppf-topology)	1.1.5
$\mathcal{G}/k$	k-groups ($= k$-group schemes)	1.1.3
$\mathcal{G}a/k$	affine k-groups	1.1.3
$a\mathcal{G}/k$	algebraic k-groups	1.1.3
$a\mathcal{G}a/k$	affine algebraic k-groups	1.1.3
$\mathcal{G}f/k$	formal k-groups	1.1.4
$d\mathcal{G}f/k$	dual-algebraic formal k-groups	1.2.1

Functors

$\widehat{F}$	completion of $F \in \mathbf{Ab}/k$	1.1.1
$\mathcal{F}_{\text{ét}}$	$= F \circ \text{red}$ étale part of $F \in \mathcal{G}f/k$	1.1.4
$\mathcal{F}_{\text{inf}}$	$= \ker\left(F \to F_{\text{ét}}\right)$ infinitesimal part of $F \in \mathcal{G}f/k$	1.1.4
$\mathbb{L}_R$	$= \mathbb{G}_{\mathrm{m}}(? \otimes R)$ linear group ass. to $R \in \mathbf{Alg}/k$	1.1.6
$\mathbb{T}_R$	$= \mathbb{L}_{R_{\text{red}}}$ torus ass. to $R \in \mathbf{Alg}/k$	1.1.6
$\mathbb{U}_R$	$= \ker\left(\mathbb{L}_R \to \mathbb{T}_R\right)$ unipotent group ass. to $R \in \mathbf{Alg}/k$	1.1.6
$\underline{\text{Pic}}_X$	Picard functor of X	2.1
$\underline{\text{Pic}}_X^0$	= identity component of $\underline{\text{Pic}}_X$	2.1
$\underline{\text{Div}}_X$	relative Cartier divisors on X	2.1
$\underline{\text{Div}}_X^0$	$= \text{cl}^{-1}\,\underline{\text{Pic}}_X^0$	2.1
$\mathcal{F}_{X,D}$	$\subset \underline{\text{Div}}_X^0$ formal group of modulus D	3.2.1

Algebraic groups

$J(C)$	Jacobian of a curve C	3.3
$J(C,D)$	Jacobian of C of modulus D	3.3
$L(C,D)$	affine part of $J(C,D)$	3.3
Pic_X	Picard scheme of X	2.1
$\mathrm{Pic}_X^{0,\mathrm{red}}$	Picard variety of X (= reduced identity component of $\underline{\mathrm{Pic}}_X$)	2.1
$\mathrm{Alb}(X)$	Albanese variety of X	2.3.1
$\mathrm{Alb}(X,D)$	Albanese variety of X of modulus D	3.2.1
$\mathrm{Alb}_{\mathcal{F}}(X)$	$= \left[\mathcal{F} \to \mathrm{Pic}_X^0\right]^\vee$ universal object for $\mathbf{Mr}_{\mathcal{F}}$	2.3.1
$\mathrm{Alb}_{\mathcal{F}}^{(1)}(X)$	universal torsor for $\mathbf{Mr}_{\mathcal{F}}$	2.3.3
$\mathrm{Alb}_{\mathcal{F}}^{(0)}(X)$	universal group for $\mathbf{Mr}_{\mathcal{F}}$	2.3.3

Chow Groups of 0-cycles

$\mathrm{CH}_0(X,D)$	relative Chow group of X of modulus D	3.4
$\mathrm{CH}_0(X,D)^0$	$= \ker\big(\deg : \mathrm{CH}_0(X,D) \to \mathbb{Z}\big)$	3.4

Rational Maps

$\mathbf{Mr}$	a category of rational maps	2.2
$\mathbf{Mr}^{\mathrm{CH}_0(X,D)^0}$	rational maps factoring through $\mathrm{CH}_0(X,D)^0$	3.4
$\mathbf{Mr}^{X,D}$	$= \big\{\varphi \mid \mathrm{mod}(\varphi) \le D\big\}$	3.2.1
$\mathbf{Mr}_{\mathcal{F}}$	$= \big\{\varphi \mid \mathrm{im}\,\tau_\varphi \subset \mathcal{F}\big\}$	2.2
τ_φ	$: L^\vee \to \underline{\mathrm{Div}}_X^{0,\mathrm{red}}$ induced transformation of φ	2.2
$\mathrm{mod}(\varphi)$	modulus of rational map φ	3.2

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